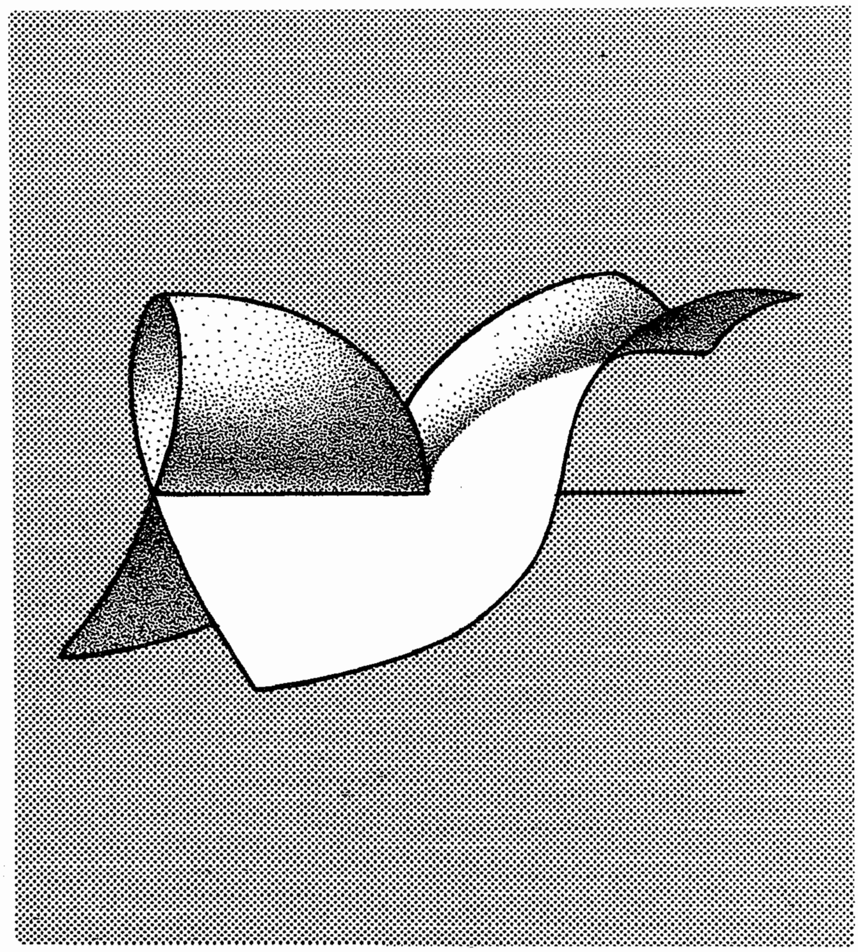
WEAKLY NORMAL SURFACE SINGULARITIES AND THEIR IMPROVEMENTS



Duco van Straten

WEAKLY NORMAL SURFACE SINGULARITIES AND THEIR IMPROVEMENTS

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR AAN DE RIJKSUNIVERSITEIT TE LEIDEN, OP GEZAG VAN DE RECTOR MAGNIFICUS, DR. J.J.M. BEENAKKER, HOOGLERAAR IN DE FACULTEIT DER WISKUNDE EN NATUURWETENSCHAPPEN, VOLGENS BESLUIT VAN HET COLLEGE VAN DEKANEN TE VERDEDIGEN OP DONDERDAG 26 FEBRUARI 1987 TE KLOKKE 15.15 UUR

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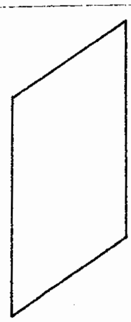
CONTENTS

aan mijn ouders

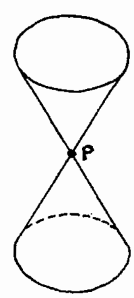
INTRODUCTION

The title of this thesis is: "Weakly Normal Surface Singularities and their Improvements". Let us try to explain this in an informal way. In the first place there is the titleword 'singularities'. Singularities are the objects of study of Singularity Theory, but unfortunately there is no definition of this last term. Let us say that Singularity Theory in the most general sense is about the interaction between the 'general' (i.e. non-singular, regular, smooth,...) and the 'special' (i.e. singular, exceptional). Fortunately the term 'surface' is more easily explained. By a surface we simply a mean two-dimensional (complex analytic) space. Such surfaces X can be obtained for example as the *zero set* of an analytic map $F : \mathbb{C}^3 \longrightarrow \mathbb{C}$, i.e. $X = F^{-1}(0)$. We can make a picture of the real part $X_{\mathbb{R}} = \{(x,y,z) \in \mathbb{R}^3 | F(x,y,z) = 0\}$, which often sheds considerable light on the nature of X. Below such pictures are shown of the simplest examples:



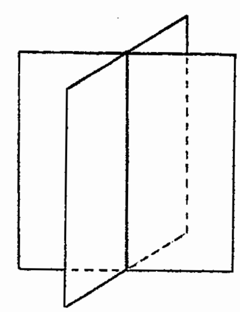
$F = z - y$

A_0



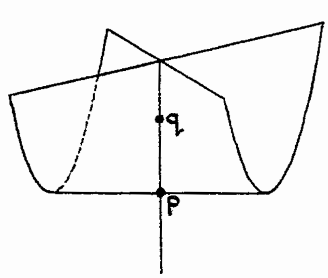
$F = z^2 - y^2 + x^2$

A_1



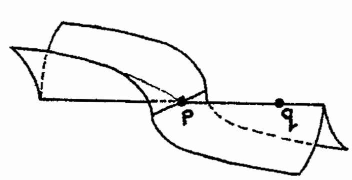
$F = z^2 - y^2$

A_∞



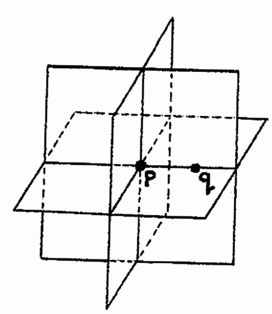
$F = z^2 - x.y^2$

D_∞



$F = z^2.y - x^3$

$Q_{\infty,\infty}$



$F = x.y.z$

$T_{\infty,\infty,\infty}$

A_0 is an example where all points look the same: there are no singular points. A_1 is the simplest example of a surface having an isolated singular point p. The other surfaces are examples of non-isolated singularities. The simplest one, A_∞, consists of two planes intersecting in a line, which is the set of singular points. In the cases D_∞ and $Q_{\infty,\infty}$ we also find a line of singular points. A difference between these two cases is that a neighbourhood of the point q for D_∞ resembles A_∞ , whereas this is not the case for $Q_{\infty,\infty}$. Finally, for $T_{\infty,\infty,\infty}$ the set of singular points is the union of the three coordinate axes, which has itself a singular point at p.

The singular points of type A_∞ , D_∞ and $T_{\infty,\infty,\infty}$ arise naturally in the following way: Take a smooth projective surface Y and embed it in $\mathbb{P}^5$. Then take a general projection $\mathbb{P}^5 \longrightarrow \mathbb{P}^3$. The image X of Y in $\mathbb{P}^3$ will then have 'ordinary singularities', i.e of type A_∞, D_∞ and $T_{\infty,\infty,\infty}$ (see [G-H], Ch. 4). A_∞ is usually called *ordinary double line* , D_∞ goes by many different names such as: *ordinary pinch point, Cayley Umbrella, Whitney Umbrella* and even sometimes *cuspidal point.* (If one refers to D_∞ as an umbrella one of course should insist on drawing the 'naked line' sticking out of the surface.) $T_{\infty,\infty,\infty}$ is usually called the ordinary triple point.

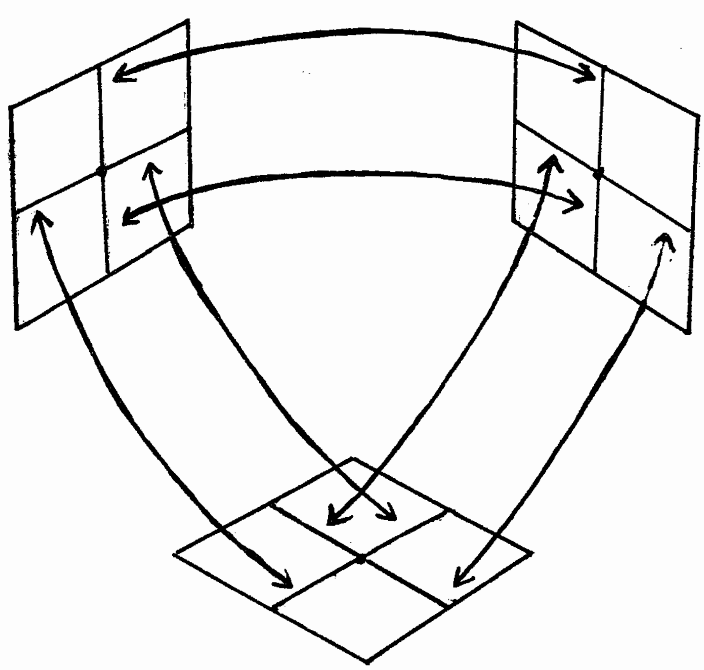
Now we have some idea as to how surface singularities may look. For surfaces X in $\mathbb{C}^3$, as above given by an equation F = 0, it is quite easy to explain the words 'normal' and 'weakly normal'. A surface X in $\mathbb{C}^3$ is normal exactly when X has only isolated singular points. A weakly normal surface X is allowed to have a one-dimensional singular locus Σ, but in a neighbourhood of a *general point* of Σ the surface X has to resemble A_∞ . Thus, all the examples mentioned, with the exception of $Q_{\infty,\infty}$, are examples of weakly normal surface singularities. The picture on the cover is another example. (Try to figure out an equation for it!)

Now we are left with the titleword 'improvements'. One could say that improvements are to weakly normal surface singularities as resolutions are to normal surface singularities. So one might well speak of 'weak resolutions' instead of 'improvements'.

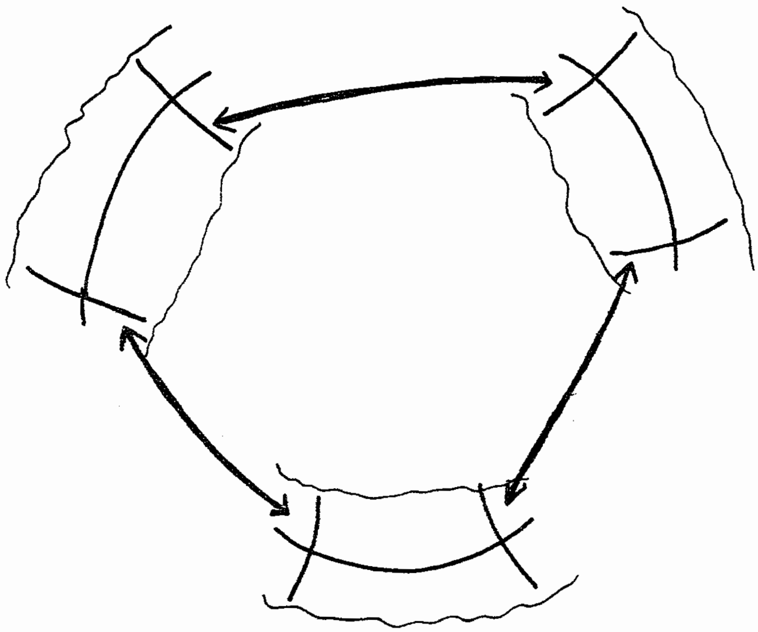
Resolution is a process which converts local data of a singularity

into global information of a smooth space. As an example, take A_1. A resolution of A_1 is obtained by *blowing up* the singular point p. In this process, the singular point p of A_1 is replaced by a projective line $\mathbb{P}^1$, lying in the blown up space, which in this case turns out to be smooth ($\approx$ cotangentspace to $\mathbb{P}^1$).
So a resolution of a normal singularity (X,p) is a smooth space Y together with a map π: $Y \longrightarrow X$, which contracts a certain compact one-dimensional set E to the singular point p.
An improvement of a weakly normal surface singularity X is something similar: it is a space Y together with a map contracting a system of curves E to the special point p. But because X can have a one-dimensional singular locus, we should not ask Y to be smooth. Instead we ask Y to have the simplest possible singularities. It turns out that in general one must allow Y to have so-called *partition singularities* which play an important role in this thesis. For example, for weakly normal X that fit into $\mathbb{C}^3$ one has to allow A_∞ and D_∞ singularities on the improving space Y.

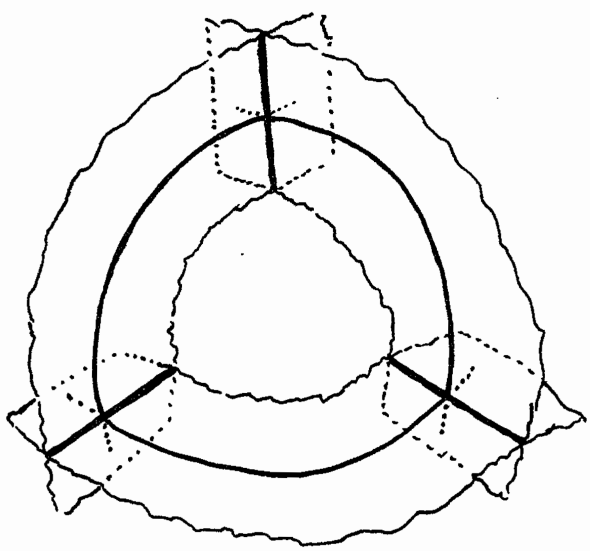
Let us give one example of an improvement: Take $X = T_{\infty,\infty,\infty}$. X can be seen as the union of three planes, glued together along coordinate axes according to the following scheme:



One can obtain an improvement by first blowing up once in each of these planes:



The advantage is that now the lines to be glued have been separated. So the improvement looks like:



We see that there are three A_∞ -singularities on the improvement Y and the map $Y \longrightarrow X$ contracts the three $\mathbb{P}^1$s that intersect cyclically.

Now that the meaning of the title has become clear one might ask the following: "What is the use of this all?" Well, in the first place it should be stressed that the improvements, just like resolutions, appear really only as a *tool* to study a singularity. These weakly normal surface singularities themselves are interesting basically for two reasons:

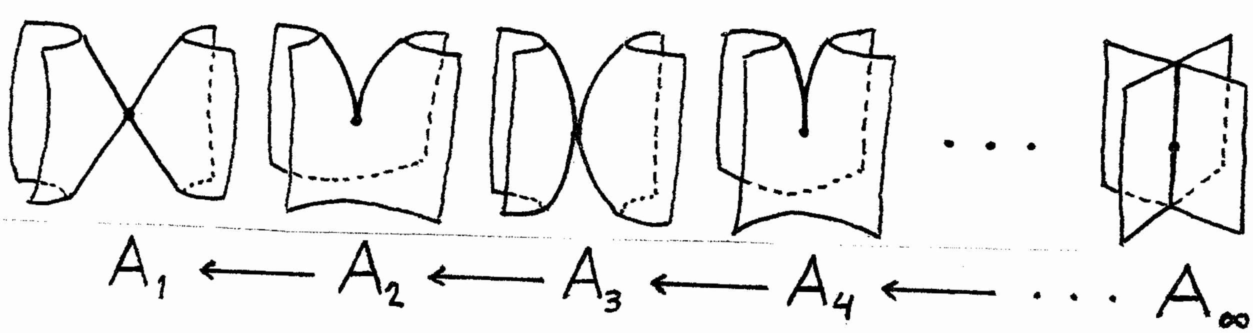
1) Because they are there.

Weakly normal surfaces form the 'simplest' class of non-isolated singularities. In the last decades there has been a considerable interest in isolated singularities. For hypersurfaces and complete intersections this has resulted in quite detailed understanding of their deformation theory and associated topology. For general normal surface singularities the theory has also reached a certain stage of perfection. A next step is to look at singularities with a one-dimensional singular locus. It is obvious that if one does so, one should first handle the case in which the structure of the singularity transverse to the singular locus is as simple as possible. For hypersurfaces and complete intersections the natural choice is to start with transverally an A_1-singularity. But if one wants to study non-isolated surface singularities it seems natural to start with the weakly normal ones, thus allowing also slightly more complicated transversal types. Although to presence of a one-dimensional singular set allows for many new phenomena to occur, it is my belief that for the weakly normal (Cohen-Macaulay) surfaces a theory can be developed to a level of detail that is comparable to the existing theory of normal surface singularities.

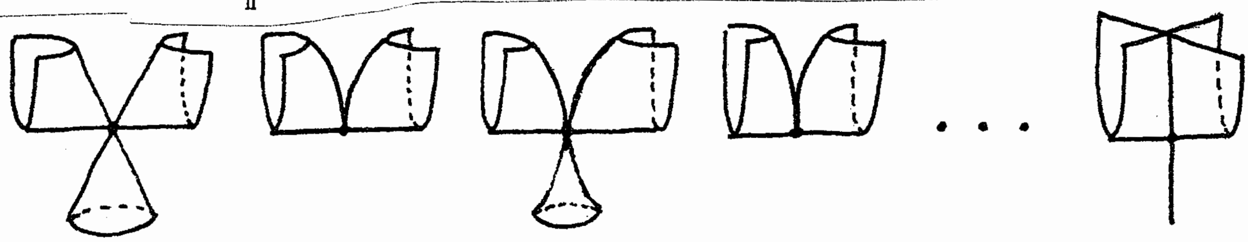
2) Because of their relation with *series of isolated singularities.*

The notion of a series of (hypersurface) singularities was introduced by Arnol'd in a sweeping phrase: "Although the series undoubtely exist, it is not at all clear what a series of singularities is." (See [Arn 1], [A-G-V], p.243.)
Let us give two examples of series:

The series A_n: $F = y.z - x^{n+1}$ $(n \geq 1)$



The series D_n: $F = z^2 - x.(y^2 - x^{n-2})$ $(n \geq 4)$



$$D_4 \longleftarrow D_5 \longleftarrow D_6 \longleftarrow D_7 \longleftarrow \cdots \quad D_\infty$$

The arrows denote the relation of *adjacency* between two singularities, which roughly corresponds to the relation *deforms into* . It is clear that the *limit* of the series A_n is A_∞ and the limit of D_n is D_∞ . Now it turns out that the *simplest* types of series that appear in the classification of isolated hypersurface singularities have as limits non-isolated singularities with a one-dimensional singular locus, transversal to which one finds an A_1-singularity. For surface singularities, not necessarily in $\mathbb{C}^3$, the phenomenon of series also occurs. It appears that the series which have a weakly normal limit can be characterized as those series for which the *geometric genus* p_g is the same for all members of the series. Although it is almost never set out explicitly in this thesis, this idea is one of our main motivations for the study of the weakly normal surface singularities.

Now that we know the subject to be of interest, we can answer the question: "What can one find in this thesis?" Of course, the Table of Contents answers this question completely, but let me give the reader a rough indication of the most important topics: Chapter 1 contains some generalities on weakly normal surfaces. Improvements are constructed with the help of a simple glueing construction. The partition singularities are introduced. Probably the most important theorem is about a very particular series of deformations to isolated singularities. Chapter 2 reviews the theory of the geometric genus. For weakly normal surfaces a geometric genus is defined and its semicontinuity under *all* flat deformations over a smooth curve germ is shown. Chapter 3 sets up the cycle theory for improvements. The most important results are the notions of *roots* and *stable models* and the realization that a satisfactory theory of the fundamental cycle is possible. Chapter 4 is called 'Applications', but contains a beginning of the study of weakly rational and minimally elliptic singularities. Further there is the classification of the so called 'Gorenstein Du Bois surfaces'. The most significant result concerns the first Betti number of a smoothing of a weakly normal space; one of the few really general results.

CHAPTER 1

WEAK NORMALITY AND IMPROVEMENTS

In this chapter we describe basic constructions and notions that will be used troughout the text, frequently without mention.
§ 1.1 introduces some properties an analytic space can have: Cohen-Macaulayness, normality and weak normality.
§ 1.2 describes some aspects of the glueing process that will be of relevance in constructing improvements.
§ 1.3 contains facts about the "building blocks" of every weakly normal surface: the partition singularities.
§ 1.4 is about resolutions and improvements.

§ 1.1 Preliminaries

By a *space* we usually mean an analytic space , sometimes an algebraic scheme over an algebraically closed field of characteristic 0. As general references we use [G-Re], [Fi] and [Ha 2]. We write X, Y, ... for our spaces and $\mathcal{O}_X$, $\mathcal{O}_Y$ for their *structure sheaves.* Most of the time we concentrate on *local properties* of X around a given point $p \in X$. We will not always make a clear distinction between the *germ* (X,p) of X at p and an appropriate *representative* of the germ (usually a sufficiently small *Stein neighbourhood* of $p \in X$). Sometimes we write " $f \in \mathcal{O}_X$ " meaning either $f \in \mathcal{O}_{X,p}$ or $f \in H^0(U,\mathcal{O}_X)$ where U is an appropriate representative of the germ (X,p).

(1.1.1) A point $p \in X$ is called a *smooth point* of X if and only if $\mathcal{O}_{X,p}$ is a regular local ring. Around such a point X looks like a smooth manifold. If this is not the case we call p a *singular point* of X. In a singular point one has a finite number of *irreducible components* coming together, each with their own *dimension* , and having as supremum dim(X,p), the dimension of X at p. A point $p \in X$ is singular precisely when $\mathrm{Embdim}(X,p) > \dim(X,p)$ where $\mathrm{Embdim}(X,p) = \dim_{\mathbb{C}}(\mathfrak{M}_p/\mathfrak{M}_p^2)$ and $\mathfrak{M}_p$

is the maximal ideal of $\mathcal{O}_{X,p}$. This *Embedding dimension* is the smallest possible dimension of a smooth germ in which the germ (X,p) can be embedded . The set Sing(X) of all singular points is an analytic set. We call it the *singular locus* and denote it usually by Σ. When X is *generically reduced* it is a proper subset of X. When $\Sigma = \{p\}$ we say that X has an *isolated singular point at p* or X *is an isolated singularity* .

The *depth* of X at p is the maximal length of a *regular sequence* in $\mathcal{O}_{X,p}$ and is denoted by depth(X,p). One always has depth(X,p) $\leq$ dim(X,p) and when equality holds we say that X is *Cohen-Macaulay* at p. Frequently we will abbreviate this to " X is CM ". The Cohen-Macaulay condition is open and invariant under *generic hyperplane section* . There exists a nice relation between depth and *local cohomology* :

$$\operatorname{depth}(X,p) \geq m \quad \Longleftrightarrow \quad \mathcal{H}^{i}_{\{p\}}(\mathcal{O}_X) = 0 \quad i=1,2,\ldots,m-1$$

(see [S-T],Thm (1.14); [Gr],Thm (3.8)) which is convenient for computational purposes. The Cohen-Macaulay condition is rather strong: it implies for instance that all irreducible components have the same dimension, and much more.

For every space there is a *dualizing complex* $\omega_X^{\cdot}$, which can be defined (locally) as $\omega_X^{\cdot} = R^{\cdot}\mathcal{H}om(\mathcal{O}_X, \Omega_Y[\dim Y])$, where $X \longrightarrow Y$ is an embedding of X into a smooth germ Y and Ω_Y is the sheaf of top differentials on Y (see [R-R]). One can show that X is CM if and only if the complex $\omega_X^{\cdot}$ reduces to a single sheaf

$$\omega_X := \omega_X^{-d} = \mathcal{E}xt^{n-d}_{\mathcal{O}_Y}(\mathcal{O}_X, \Omega_Y) \qquad (n = \dim(Y),\ d = \dim(X))$$

For a CM-space X one defines the *type* , *Cohen-Macaulay type* or *Gorenstein type* to be the number

$$\operatorname{type}(X,p) := \dim_{\mathbb{C}}(\omega_{X,p}/\mathfrak{m}_{X,p}\cdot\omega_{X,p})$$

If the type is 1 we say that X is *Gorenstein* at p. Again, this is an open condition and is preserved under generic hyperplane section. So X is Gorenstein iff $\omega_{X,p} \approx \mathcal{O}_{X,p}$.
Examples of spaces which are Gorenstein are all *hypersurface singularities* or more generally *complete intersections* (see [Lo]).

(1.1.2) Besides Cohen-Macaulayness, there is another kind of neatness property a space can have: *normality.*

On a *reduced* space X one defines a sheaf $\tilde{\mathcal{O}}_X$ consisting of all bounded holomorphic functions on $X-\Sigma$. Algebraically, $\tilde{\mathcal{O}}_X$ can be characterized as the integral closure of $\mathcal{O}_X$ in its total quotient ring, the ring of meromorphic functions on X. It is a very fundamental fact that $\tilde{\mathcal{O}}_X$ is a *coherent sheaf of* $\mathcal{O}_X$ - *algebras* (see [Na], [G-Re]). Hence one can define a space

$$\tilde{X} = \mathrm{Specan}(\tilde{\mathcal{O}}_X)$$

called the *normalization* of X. The natural inclusion

$$\mathcal{O}_X \subseteq \mathcal{O}_{\tilde{X}} = \tilde{\mathcal{O}}_X$$

gives rise to a *finite map* $n : \tilde{X} \longrightarrow X$ called the *normalization (mapping)*. When the map n is an isomorphism we say that X is *normal* . So a space X is normal when the Riemann extension theorem holds for X. For a normal space the codimension of Σ in X is ≥ 2 and conversely when this is the case , normality of X is equivalent to the *cohomological condition*

$$\mathcal{H}^1_\Sigma(\mathcal{O}_X) = 0$$

(because then the boundedness of holomorphic functions on $X-\Sigma$ is then automatic). So a normal space of dimension 1 is smooth and an isolated singularity of which all irreducible components have dimension ≥ 2 is normal iff it has depth ≥ 2. The normalization map $n : \tilde{X} \longrightarrow X$ has the following *universal property* :

> A map $f : Y \longrightarrow X$ with Y *normal* and $f(Y) \not\subset \Sigma$ can be factorized through n .

Because we add to the structure sheaf functions defined *outside* Σ which need not extend to functions on X (let alone continuous ones) normalization can be a quite drastic operation. The irreducible components of X become separated and every part of Σ in codimension 1 has to disappear. In other words, the underlying topological space of X is altered drastically.

(1.1.3) A way to avoid this change in topology is clear: add only *continuous* functions to the structure sheaf. This leads to the notion of ***weak normality***, as introduced by Andreotti and Norguet in [A-N]. One defines a sheaf $\bar{\mathcal{O}}_X$ consisting of all continuous functions which are holomorphic on $X-\Sigma$. Again $\bar{\mathcal{O}}_X$ is a sheaf of coherent $\mathcal{O}_X$ - algebras and one can form the space

$$\bar{X} = \mathrm{Specan}(\bar{\mathcal{O}}_X)$$

called the ***weak normalization*** of X. We get a finite analytic mapping $w : \bar{X} \longrightarrow X$, which by construction is a *homeomorphism* of the underlying topological spaces. This weak normalization mapping w has the following universal properties (see [Fi]):

1) w factorizes through every mapping $Y \longrightarrow X$, which is a homeomorphism of underlying spaces.
2) Every mapping $Y \longrightarrow X$, with Y weakly normal, factorizes through w .

Because of the first property the weak normalization is sometimes called the *maximalization.* A space for which w is an isomorphism is called *weakly normal.* Often we will abbreviate this to "X is WN". Normality and weak normality are open conditions and preserved under generic hyperplane sections (for weak normality this last statement is not so easy , because there is no obvious cohomological way to formulate weak normality (see [A-L], [Vi])), but contrary to the Cohen-Macaulay and Gorenstein conditions, this is in general not true if we require the hyperplane to pass through a special point $p \in X$.

As normalization, weak normalization can be done algebraically (see [A-B]), but in charateristic $\neq 0$ it gives rise to two different notions: weak normality and semi normality (see [Ad 1], [C-M], [G-T], [Ma]).

(1.1.4) We mention two further operations to make a space "better" :

1) we can give a space X *depth* ≥ 1 *at* p by changing the structure sheaf $\mathcal{O}_X$ to $\mathcal{O}_X/\mathcal{H}^0_{\{p\}}(\mathcal{O}_X)$.
2) we can give a space X *depth* ≥ 2 *at* p by first throwing away the one-dimensional components and then changing the structure

sheaf to $i_* i^* \mathcal{O}_X$ where $i : X-\{p\} \longrightarrow X$ is the inclusion map. There seems to be no canonical way to give a space X higher depth.

By a *curve* we will mean a (germ of a) space of dimension 1. By a *surface* we will mean a (germ of a) space *purely* of dimension 2, i.e having only two-dimensional components. By the above procedure we can define for curves and surfaces an operation of *Cohen-Macaulification* $c : \hat{X} \longrightarrow X$, which is sometimes useful. Finally, we define the normalization and weak normalization for general X by taking the normalization or weak normalization of its *reducuction* X_{red}.

The general structure of a WN - curve is easy to describe:

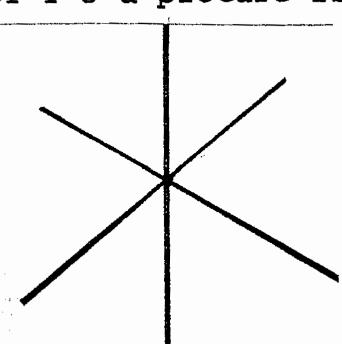
(1.1.5) Proposition : A germ (X,p) of a *weakly normal curve* is isomorphic to the germ of r lines in general position in $\mathbb{C}^r$, where r is the number of irreducible components of X at p . So one has:

$$\mathcal{O}_{X,p} \approx \mathbb{C}\{Y_1,\ldots,Y_r\}/(Y_i.Y_j\ ;\ i\neq j)$$

Further: $\mathrm{Mult}(X,p) = r$, $\mathrm{type}(X,p) = r-1$ $(r\geq 3)$.

proof : This is a nice exercise. □

This curve singularity is called L_r^r (see [B-G]). For r=1 we have a single smooth branch, for r=2 we have an ordinary node (which also goes by the name A_1). For r=3 a picture is shown below.



(1.1.6) Remark : The curve germs L_r^r can be characterized alternatively as the unique reduced curve germs (C,p) for which one has the equality $\delta(C,p) = r - 1$, where $\delta(C,p)$ is the delta-invariant (see (1.2.23)) and r the number of

irreducible components of the germ (C,p).
A surface germ (X,p) which is both WN and CM is as close as possible to a *normal* germ without necessarily being so. This important class of spaces will be the main object of our study.

(1.1.7) Definition : Let (X,p) be a germ of an analytic space. We say that:

1) (X,p) is AWN (almost weakly normal) if and only if $X - \{p\}$ is weakly normal.
2) (X,p) is WNCM if and only if X is weakly normal and Cohen-Macaulay.

(1.1.8) Lemma : Let X be (an appropriate representative of) a *surface* germ. Let Σ be its singular locus and p the special point. Then (X,p) is AWN if and only if X has the following normal forms around a point q:

$$q \in X - \Sigma \qquad \mathcal{O}_{X,q} \approx \mathbb{C}\{Y_1, Y_2\}.$$

$$q \in \Sigma - \{p\} \qquad \mathcal{O}_{X,q} \approx \mathbb{C}\{Z, Y_1, \ldots, Y_r\}/(Y_i . Y_j; i \neq j)$$

proof : A hyperplane section trough a general point q of Σ is weakly normal, so from (1.1.5) it follows that it is (locally) isomorphic to L_r^r for some r (which may depend on the irreducible component of Σ under consideration). As every topologically trivial family in which L_r^r appears is in fact analytically trivial (see [B-G]), we get the above normal form. ∎

(1.1.9) Remark : Let (X, Σ, p) be a germ of a WNCM-surface with singular locus Σ. Then exactly one of the following things is true:

A. $\dim(\Sigma) = 0$. Then $\Sigma = \{p\}$ is the unique singular point of X. X is then normal (and hence irreducible).

B. $\dim(\Sigma) = 1$. Then Σ is (a germ of) a curve with possibly p as singular point. The irreducible components of X intersect each other in components of Σ ($X - \{p\}$ is connected).

The following lemma is convenient for proving that an AWN - space is in fact WN:

(1.1.10) Lemma : Let (X,p) be a germ of an analytic space. If $X-\{p\}$ is WN and $\operatorname{depth}(X,p) \geq 2$ then X itself is WN.

proof : If f is a continuous function on X, holomorphic outside the singular set Σ, then it follows from the WN property of $X-\{p\}$ that f is in fact in $\mathcal{O}_{X-\{p\}}$, so by depth ≥ 2 it follows that $f \in \mathcal{O}_X$. ∎

(1.1.11) Remark : We refer to [A-H], lemma (16.1) for a general "weakly normal extension" theorem.

(1.1.12) Corollary : A hypersurface $X \subset \mathbb{C}^3$ is WN precisely when transverse to the singular locus we find generically an A_1- singularity (or X has isolated singularities).

(1.1.13) Remark : General hypersurfaces with a one-dimensional singular locus, transverse to which one finds generically an A_1- singularity, were studied recently in [Pe],[Sie 1] and [Sie 2].

All these hypersurface examples of course are Gorenstein.

(1.1.14) Lemma : Let (X,p) be a Gorenstein WNCM-surface germ. Then transverse to the singular locus we find generically an A_1-singularity (or X has an isolated singular point).

proof : This is clear, because the Gorenstein property is preserved under hyperplane section, and L_r^r is a Gorenstein curve only when $r=1$ or 2. ∎

(1.1.15) Convention : From now on, when we refer to a surface germ by writing (X,Σ,p), we will explicitly mean that $\dim(\Sigma,p) = 1$, in order to avoid special mention of the case that (X,p) is an isolated singularity.

The converse of (1.1.14) is certainly not true. We give an example that will play a role in the sequel. (It also appears in [G-T].)

(1.1.16) Example : Let X be the surface germ at the origin in $\mathbb{C}^4$, given by the following system of equations:

$$\operatorname{rank}\begin{pmatrix} x & y & z \\ z & w & x.y \end{pmatrix} \le 1$$

where x,y,z and w are coordinates on $\mathbb{C}^4$.
This space has two irreducible components, given by the following systems of equations:

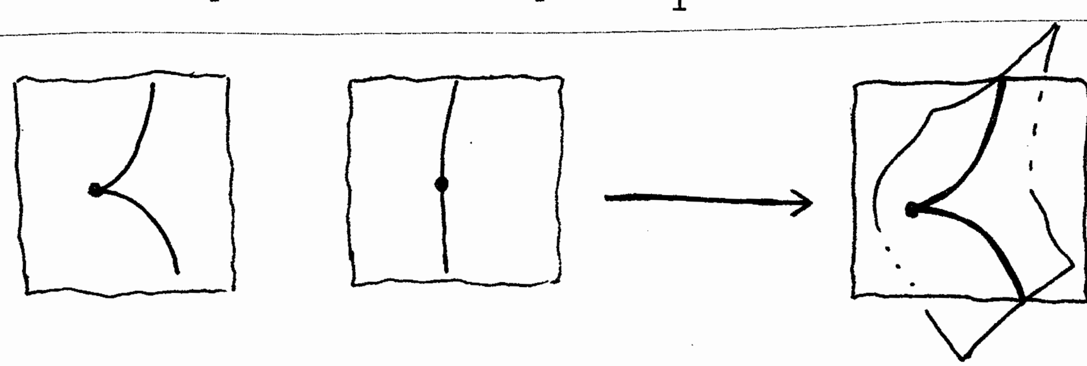
$X_1 \quad : \quad x = z = 0$

$X_2 \quad : \quad \operatorname{rank}\begin{pmatrix} x & y & z \\ z & w & x.y \end{pmatrix} \le 1 \quad , \quad y^3 = w^2$

The second component can more easily be given parametrically as the image of $\mathbb{C}^2$ under the mapping :

$$(s,t) \longrightarrow (s,t^2,s.t,t^3)$$

Note that this component is not Cohen - Macaulay.
The singular locus is $X_1 \cap X_2$, so is given by the equations $x = z = y^3-w^2= 0$. It is easy to see that outside 0 X_1 and X_2 are transverse, so the transversal type is A_1. By the general theory of *determinantal singularities* (see also § 1.3) it follows that X is CM and has type 2 , so is non-Gorenstein. We see that the normalization of X consists of two disjoint copies of $\mathbb{C}^2$, called $\tilde{X}_1$ and $\tilde{X}_2$. The inverse image of Σ under the normalization map will be called $\tilde{\Sigma}$ and consists of two components: one smooth curve in $\tilde{X}_2$ and a component with a cusp in $\tilde{X}_1$:



Conversely, we can find X by glueing $\tilde{X}_1$ to $\tilde{X}_2$ along Σ.

§ 1.2 Glueing

A fundamental problem is the following: When we consider a space X and an *equivalence relation* R on X, under what conditions can we give the set of equivalence classes the structure of a space ? Here one should think of "space" in some categorial sense, like analytic spaces, algebraic spaces or schemes, and the quotient X/R is required to possess a universal property within the category. In general this problem is difficult and not many results are known (at least to me).

In the category of ringed spaces one always can form the quotient by putting $\mathcal{O}_{X/R} = \pi_*\mathcal{O}_X$, where $X \xrightarrow{\pi} X/R$ is the quotient map. In the category of (reduced) analytic spaces there is one simple and often useful criterion in the case of *proper* equivalence relations (i.e. in the case that $X \longrightarrow X/R$ is a proper map) due to H. Cartan (see [Ca]):

(1.2.1) Theorem : Let X be an analytic space and R a proper equivalence relation on X.

Then $(X/R, \mathcal{O}_{X/R})$ is an analytic space if and only if $\mathcal{O}_{X/R}$ locally seperates the points of X/R (i.e. for all $x \in X/R$ there is an open $\mathcal{U} \ni x$ such that for all $y, z \in \mathcal{U}$ there is an $f \in H^0(\mathcal{U}, \mathcal{O}_{X/R})$ with $f(y) \neq f(z)$).

A special case of the quotient construction is the formation of the *push-out* or *fibered sum* of two spaces. In that case two maps $p: T \longrightarrow Y$ and $q: T \longrightarrow Z$ are given and we ask for a space $X := Y \coprod_T Z$ making a commutative

(1.2.2) Diagram:

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left] & \\
Z \arrow[r] \arrow[drr, bend right] & X = Y \coprod_T Z \arrow[dr, "!"] & \\
& & W
\end{tikzcd}$$

having the universal property as suggested by the diagram.
The situation we have in mind is the following: Consider a space $\tilde{X}(=Y)$ and a *subspace* $\tilde{\Sigma}$ $(=T) \longrightarrow \tilde{X}$ together with a *finite surjective map* $\tilde{\Sigma} \longrightarrow \Sigma$ $(=Z)$. The push-out space X we call *the space obtained from* $\tilde{X}$ *by glueing* $\tilde{\Sigma}$ *to* Σ.
This situation in particular arises when we consider the normalization $\tilde{X} \xrightarrow{n} X$ of a space X and put $\Sigma = \mathrm{Sing}(X)$, $\tilde{\Sigma} = n^{-1}(\Sigma)$.
On the level of structure sheaves we have to consider a diagram dual to (1.2.2). For simplicity we assume that the natural map $\mathcal{O}_\Sigma \longrightarrow n_*\mathcal{O}_{\tilde{\Sigma}}$ is an *inclusion*, and we suppress the n_* from the notation. Thus we write $\mathcal{O}_\Sigma \leq \mathcal{O}_{\tilde{\Sigma}}$ and tacitly consider all sheaves as living on X.

(1.2.3) Diagram:

$$\begin{tikzcd}
\mathcal{J} \arrow[r, hook] \arrow[d, "\wr"] & \mathcal{O}_X \arrow[r, two heads] \arrow[d, hook] & \mathcal{O}_\Sigma \arrow[d, hook] \\
\mathcal{J} \arrow[r, hook] & \mathcal{O}_{\tilde{X}} \arrow[r, two heads] \arrow[d, two heads] & \mathcal{O}_{\tilde{\Sigma}} \arrow[d, two heads] \\
& \mathcal{G} \arrow[r, "\sim"] & \mathcal{G}
\end{tikzcd}$$

In this diagram $\mathcal{J} = \ker(\mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_{\tilde{\Sigma}})$ is the ideal sheaf of $\tilde{\Sigma}$ and $\mathcal{G} = \mathcal{O}_{\tilde{X}}/\mathcal{O}_X = \mathcal{O}_{\tilde{\Sigma}}/\mathcal{O}_\Sigma$ is a certain $\mathcal{O}_\Sigma$ - module which we call the *glueing module* of the situation.

The corresponding situation for the (local) rings is handled by:

(1.2.4) Lemma : Consider the pull-back diagram of rings:

$$\begin{tikzcd}
I \arrow[r, hook] \arrow[d, "\wr"] & R \arrow[r, two heads] \arrow[d, hook] & S \arrow[d, hook] \\
I \arrow[r] & \tilde{R} \arrow[r, two heads, "\pi"] & \tilde{S}
\end{tikzcd}$$

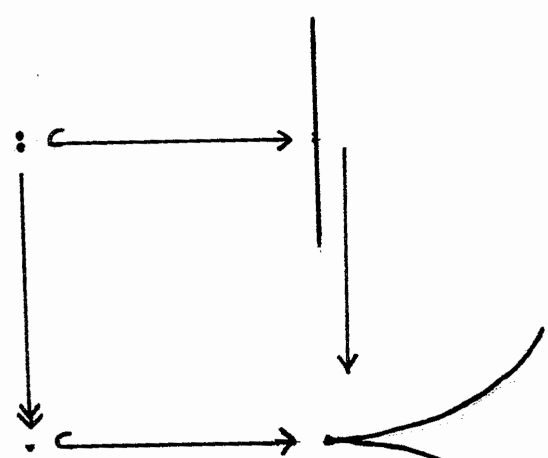
Then, if $\tilde{S}$ is a finitely generated S-module, then $\tilde{R}$ is a finitely generated R-module.

proof : This is trivial, but let us spell out the proof. Let $\tilde{\psi}_1 = 1$, $\tilde{\psi}_2,\ldots,\tilde{\psi}_k$ be generators of $\tilde{S}$ as S-module. Lift them to elements $\psi_1 = 1$, $\psi_2,\ldots,\psi_k \in \tilde{R}$. Let $\tilde{r} \in \tilde{R}$. As the $\tilde{\psi}_i$ generate $\tilde{S}$ over S, we can write $\pi(\tilde{r}) = \sum \tilde{\psi}_i . s_i$, $s_i \in S$. Choose elements r_i such that $\pi(r_i) = s_i$. Then $\pi(\tilde{r} - \sum \psi_i . r_i) = 0$, hence $\tilde{r} - \sum \psi_i . r_i \in I \subset R$. So $\tilde{r} = \psi_1 . (\tilde{r} - \sum \psi_i . r_i) + \sum_{i=2}^{k} \psi_i . r_i$, hence the ψ_i generate $\tilde{R}$ over R. ∎

(1.2.5) **Corollary :** In the above situation, assume that $\tilde{S}$ is a finitely generated S-module. Then if $\tilde{R}$ is a finitely generated k-algebra or a (semi-) local analytic ring, then the same is true for the ring R. The argument is well known: $\tilde{R}$ is finite over the subring of R generated by the coefficients of the integrality equations of the algebra generators of $\tilde{R}$ and this subring is noetherian and R is finite over this subring.

(1.2.6) **Conclusion :** In the category of analytic spaces (or schemes of finite type over a field k) we can form the push-out as in diagram (1.2.2) if p is an inclusion and q is a finite map. The resulting map Y $\longrightarrow$ X is finite.

(1.2.7) **Example :** Take a line and identify two infinitesimally near points.



This corresponds to the following diagram of rings:

$$\begin{tikzcd} (x^2,x^3) \arrow[r, hook] \arrow[d, "\wr"] & \mathbb{C}[x^2,x^3] \arrow[r, two heads] \arrow[d, hook] & \mathbb{C} \arrow[d, hook] \\ (x^2) \arrow[r, hook] & \mathbb{C}[x] \arrow[r, two heads] & \mathbb{C}[x]/(x^2) \end{tikzcd}$$

So indeed the push-out has a cusp as singular point. (c.f.[Se],p.70)

(1.2.8) Example : Take a line in a plane and identify this line to a point. In this case the conditions of (1.2.6) are not fulfilled, because the line is not mapped finitely to the point and indeed the push-out does not exist as an analytic space.

In practice it will be useful to have an explicit set of generators for the pull-back ring R. We consider the case that $\tilde{R}$ and $\tilde{S}$ are (semi-) local analytic rings and $\tilde{S}$ is module-finite over a local analytic ring $S = \mathbb{C}\{\tilde{\varphi}_i\}$. We lift the elements $\tilde{\varphi}_i$ to elements $\varphi_i \in \tilde{R}$ and take as in (1.2.4) liftings ψ_j of generators of $\tilde{S}$ over S. Finally, we let f_k be generators of the ideal I (all except $\psi_1=1$ assumed to be in the Jacobson radical). For simplicity of notation we suppress the index ranges.

(1.2.9) Lemma : Consider the local analytic algebra

$$\bar{R} = \mathbb{C}\{\varphi_i, \psi_j \cdot f_k\}$$

Then: $R = \bar{R}$

proof : It is clear that $\bar{R} \subset R$. Let $x \in R$. Then we can write $x = \sum a_i \cdot f_i + P(\varphi_j)$, with $a_i \in \tilde{R}$, and P analytic in the φ_j's, so $P(\varphi_j) \in \bar{R}$. Now we can expand the a_i's :

$a_i = \sum b_{ij} \cdot f_j + \sum Q_{ik}(\varphi) \cdot \psi_k$, with $b_{ij} \in \tilde{R}$, and $Q_{ik} \in \bar{R}$.

When we substitute this in the above expression we get:

$x = \sum b_{ij} \cdot f_i \cdot f_j + \sum Q_{ik}(\varphi) \cdot f_i \cdot \psi_k + P(\varphi)$.

The last two parts are in the ring $\bar{R}$, and continuing this way we see that one has the following approximation property:

$$R = \bigcap_{n=1}^{\infty}(\bar{R} + I^n)$$

When we know that R is a finite $\bar{R}$-module, we are finished by the noetherianity of $\bar{R}$. Consider still another ring:

$$A := \mathbb{C}\{\varphi_i, f_k\} \leq \bar{R} \leq R \leq \tilde{R}$$

Now by the preparation theorem (see[Na],thm 1), a module M over A is finitely generated if and only if $M/\mathfrak{m}.M$ is of finite dimension over $A/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal (φ_i, f_k) of the ring A. So $\tilde{R}$ is finite as A-module if and only if $\tilde{R}.\mathfrak{m}$ contains a power of the Jacobson radical. But in our situation this is clear, because $\tilde{S}.(\varphi_i)$ contains a power of its radical, as $\tilde{S}$ is finite over S. ∎

(1.2.10) Remark : In the case of finitely generated algebras over a field k, it really can happen that R is <u>not</u> a finite module over $\bar{R}$, so in that case (1.2.9) is not true. One needs some kind of completion of $\bar{R}$ along I.

Let us redo example (1.1.16) by glueing and using (1.2.9): functions on $\tilde{X} = \tilde{X}_1 \amalg \tilde{X}_2$ are pairs of functions $(f,g) \in \mathcal{O}_{\tilde{X}_1} \times \mathcal{O}_{\tilde{X}_2} = \mathbb{C}\{u,v\}\times\mathbb{C}\{s,t\}$.

Generators for $\mathcal{I}$: $(u^2 - v^3, 0)$, $(0,s)$

Generators for $\mathcal{O}_\Sigma$: (u,t^3) , (v,t^2)

Generators for $\mathcal{O}_{\tilde{\Sigma}}$ as $\mathcal{O}_\Sigma$- module : $(0,t)$, $(1,1)$

So by (1.2.9) generators of the local ring $\mathcal{O}_X$ of the glueing space are:

(u,t^3)	(v,t^2)	$(u^2 - v^3,0)$	$(0,s)$	$(0,s.t)$
w	y		x	z

The middle element $(u^2 - v^3, 0)$ is redundant, as one has the following identity: $(u^2 - v^3, 0) = (u,t^3)^2 - (y,t^2)^3$

It is not difficult to show that the ideal of relations between the elements x,y,z,w is the ideal of 2×2-minors of the matrix of (1.1.16), so the space obtained by glueing is the same we started with.

Given any finite mapping $\tilde{X} \xrightarrow{n} X$ between two spaces, which are generically isomorphic, there is a canonical way to consider X as obtained from $\tilde{X}$ by glueing along a subspace.

(1.2.11) Definition : Let $n : \tilde{X} \longrightarrow X$ be a mapping. The *conductor* of n is the $\mathcal{O}_X$ ideal(sheaf) $\mathcal{C}$:

$$\mathcal{C} := \mathrm{Ann}_{\mathcal{O}_X}(n_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X) = \{g \in \mathcal{O}_X \mid g.\mathcal{O}_{\tilde{X}} \subseteq \mathcal{O}_X\} = \mathcal{H}om_{\mathcal{O}_X}(n_*\mathcal{O}_{\tilde{X}},\mathcal{O}_X)$$

For a finite map between reduced spaces we always consider $\mathcal{O}_X \subseteq \mathcal{O}_{\tilde{X}}$ and then $\mathcal{C}$ is also an ideal in $\mathcal{O}_{\tilde{X}}$. $\mathcal{C}$ can be characterized as the biggest $\mathcal{O}_X$-ideal which is also an $\mathcal{O}_{\tilde{X}}$-ideal.
In this situation we put:

$$\Sigma := \mathrm{supp}(\mathcal{O}_\Sigma), \qquad \mathcal{O}_\Sigma := \mathcal{O}_X/\mathcal{C}$$

$$\tilde{\Sigma} := \mathrm{supp}(\mathcal{O}_{\tilde{\Sigma}}), \qquad \mathcal{O}_{\tilde{\Sigma}} := \mathcal{O}_{\tilde{X}}/\mathcal{C}$$

We call Σ and $\tilde{\Sigma}$ the *glue-loci* with their *conductor structure*.
It is now a *tautology* that X can be considered as obtained from $\tilde{X}$ by glueing $\tilde{\Sigma}$ to Σ, i.e. we have a diagram as (1.2.3).

(1.2.12) Lemma : Let $\tilde{X} \xrightarrow{n} X$ be the normalization map. When $\mathrm{depth}(X,p) \geq 2$, then $\mathrm{depth}(\Sigma,p) \geq 1$ and $\mathrm{depth}(\tilde{\Sigma},q) \geq 1$, where Σ, $\tilde{\Sigma}$ are as above and $q = n^{-1}(p)$.

proof : This is because the conductor is a $\mathcal{H}om$, so has depth ≥ 2 as soon as X has (see [Schl], lemma 1). ∎

(1.2.13) Lemma : Let X be a reduced analytic space and let $\tilde{X} \xrightarrow{n} X$ be the normalization mapping. If X is weakly normal, then $\mathcal{C} \subset \mathcal{O}_{\tilde{X}}$ is a radical ideal, so $\tilde{\Sigma}$ and Σ are reduced.

Proof : Let $f \in \mathrm{rad}(\mathcal{C})$, then $f_{|\tilde{\Sigma}} = 0$. But then $g.f_{|\tilde{\Sigma}} = 0$ and functions vanishing on Σ certainly descend to continuous functions on X, hence $f.\mathcal{O}_{\tilde{X}} \subset \mathcal{O}_X$. So $f \in \mathcal{C}$, giving $\mathcal{C} = \mathrm{rad}(\mathcal{C})$. ∎

(1.2.14) Remark : The converse of (1.2.13) is unfortunately not true: the conductor can very well be reduced without X being weakly normal. An example is

$\mathcal{O}_X = \mathbb{C}\{y,\ x.y,\ x^2.y,\ x^3.y,\ x^4,\ x^6\} \subset \mathbb{C}\{x,\ y\} = \mathcal{O}_{\tilde{X}}$.

The conductor is the ideal (y) which is clearly a radical ideal, but $x^2 \notin \mathcal{O}_X$ so X is not WN.

(1.2.15) Remark : Given a mapping $Y \longrightarrow X$ one can look at the *canonical equivalence relation* $R := Y \times_X Y \subset Y \times Y$ induced by this map. For a WN space and $\tilde{X} \longrightarrow X$ the normalization map, the space R is reduced (see [A-H],(15.1.1)) and in the example under (1.2.14) R is not reduced. However, there are spaces which are <u>not</u> WN but have reduced R. An example is

$$X = \{(x,y,z) \in \mathbb{C}^3 \mid z^3 - x.y^3 = 0\}$$

It can be shown (see[Ma]) that the weak normalization $\bar{X}$ can be obtained as the quotient of its normalization $\tilde{X}$ by the <u>reduction</u> R_{red} of the canonical equivalence relation R. So it may very well happen that the quotient of $\tilde{X}$ with respect to the canonical equivalence relation R obtained from $\tilde{X} \longrightarrow X$, is not isomorphic to X (!). This is the reason why I prefer to glue with the conductor. One can ask whether it is true that when X has a reduced conductor, then X is WN around a generic point of Σ.

Let us now consider the question of Cohen-Macaulayness of the push-out space X. This can be discussed conveniently with the use of diagram (1.2.3). We study the situation around $p \in X$. We take local cohomology of (1.2.3). This gives a diagram with exact rows:

(1.2.16) Diagram :

$$\begin{tikzcd}
{} \arrow[r] & H^i_{\{p\}}(\mathcal{O}_X) \arrow[r] \arrow[d] & H^i_{\{p\}}(\mathcal{O}_{\tilde{X}}) \arrow[r] \arrow[d] & H^i_{\{p\}}(\mathcal{G}) \arrow[r] \arrow[d, "\wr"] & H^{i+1}_{\{p\}}(\mathcal{O}_X) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H^i_{\{p\}}(\mathcal{O}_\Sigma) \arrow[r] & H^i_{\{p\}}(\mathcal{O}_{\tilde{\Sigma}}) \arrow[r] & H^i_{\{p\}}(\mathcal{G}) \arrow[r] & H^{i+1}_{\{p\}}(\mathcal{O}_\Sigma) \arrow[r] & {}
\end{tikzcd}$$

(1.2.17) Proposition : Let
$$\begin{tikzcd} \tilde{\Sigma} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{X} \arrow[d] \\ \Sigma \arrow[r] & X \end{tikzcd}$$
be a glueing diagram

Assume that $\tilde{X}$ is CM of dimension $n \geq 2$, that $\tilde{\Sigma}$ and Σ are CM and that $\mathrm{supp}(\mathcal{G})=\Sigma\neq\emptyset$. Then equivalent are:

1) X is CM

2) $\mathcal{G}$ is a maximal CM $\mathcal{O}_\Sigma$-module and $\dim(\Sigma) = n - 1$

3) $H^{n-2}_{\{p\}}(\mathcal{G})=0$ and $\dim(\Sigma) = n - 1$

proof : If $\tilde{X}$ is CM of dimension n, then $H^i_{\{p\}}(\mathcal{O}_{\tilde{X}})=0$, $i=0,\ldots,n-1$. From the top row of (1.2.16) we then see:

$$H^i_{\{p\}}(\mathcal{O}_X) \approx H^{i-1}_{\{p\}}(\mathcal{G}),\ i=1,\ldots,n-1,\ (H^0_{\{p\}}(\mathcal{O}_X)=0)$$

so X is CM precisely when $H^i_{\{p\}}(\mathcal{G})=0$ for $i=0,\ldots,n-2$. This implies that $\dim(\Sigma)=n-1$, because $\mathrm{supp}(\mathcal{G})\neq\emptyset$. (By [Gr], prop. 6.4 one always has $H^k_{\{p\}}(\mathcal{G}) \neq 0$ with $k = \dim(\mathrm{supp}(\mathcal{G}))$.) and hence $\mathcal{G}$ is a maximal CM $\mathcal{O}_\Sigma$-module, proving 1) $\Rightarrow$ 2).
The implication 2) $\Rightarrow$ 3) is trivial.
If Σ and $\tilde{\Sigma}$ are CM of dimension $n - 1$, then it follows from the bottom row of (1.2.16) that $H^i_{\{p\}}(\mathcal{G})=0$, $i=0,\ldots,n-3$, so the condition $H^{n-2}_{\{p\}}(\mathcal{G})$ gives the implication 3) $\Rightarrow$ 1). ■

(1.2.18) Remark : There is one noteworthy case in which the condition $H^{n-2}_{\{p\}}(\mathcal{G})=0$ of the above proposition is automatic: the case in which the exact sequence

$$0 \longrightarrow \mathcal{O}_\Sigma \longrightarrow \mathcal{O}_{\tilde{\Sigma}} \longrightarrow \mathcal{G} \longrightarrow 0$$

<u>splits</u> as a sequence of $\mathcal{O}_\Sigma$-modules. Then the associated long exact local cohomology sequence of (1.2.16) splits into short exact sequences, and thus from the fact that Σ and $\tilde{\Sigma}$ are CM one can draw the conclusion that $H^{n-2}_{\{p\}}(\mathcal{G})$.
In particular, if Σ is normal this is true, as in that case the above sequence is split by the trace map

$$\mathcal{O}_{\tilde{\Sigma}} \xrightarrow{\mathrm{tr}} \mathcal{O}_\Sigma.$$

(1.2.19) Corollary : Let (X,p) be a *surface* germ, assumed to be *pure* and *reduced*.

Let
$$\begin{array}{ccc} \tilde{\Sigma} & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ \Sigma & \longrightarrow & X \end{array}$$
be the glueing diagram of the normalization map

$\tilde{X} \xrightarrow{n} X$. Then X is CM if and only if:

$$H^0_{\{p\}}(\mathscr{C})=0, \qquad \mathscr{C} = \mathcal{O}_{\tilde{X}}/\mathcal{O}_X = \mathcal{O}_{\tilde{\Sigma}}/\mathcal{O}_{\Sigma}.$$

If X is CM, then $\tilde{\Sigma}$ and Σ are CM-curves.

proof : The first statement is just a special case of (1.2.17). The second statement follows from (1.2.12). ∎

Although reduced conductor is not enough to conclude weak normality, there is a description of WNCM surface germs (X,Σ,p):

(1.2.20) Theorem : Let (X,Σ,p) be a surface germ. Let $\tilde{X} \xrightarrow{n} X$ the normalization and $\tilde{\Sigma}=n^{-1}(\Sigma)$. Then equivalent are:

1) X is WNCM.

2) $\tilde{X}$ is purely two-dimensional, $\tilde{\Sigma}$ and Σ are reduced curves, and $H^0_{\{p\}}(\mathscr{C})=0$.

proof : 1) $\Rightarrow$ 2): This is a combination of (1.2.13) and (1.2.19)
2) $\Rightarrow$ 1): The fact that X is CM follows from (1.2.19).
By (1.1.10), to prove that X is WN, it is sufficient to prove that $X-\{p\}$ is WN. But away from p, we may assume that the map $\tilde{\Sigma} \longrightarrow \Sigma$ is unbranched. As $\tilde{\Sigma}-n^{-1}(p) \subset \tilde{X}-\mathrm{Sing}(\tilde{X})$, we have a standard situation above $x \in \Sigma-\{p\}$, giving after a simple computation the normal form of (1.1.8). ∎

We now give another interpretation of the condition "$H^0_{\{p\}}(\mathscr{C})=0$" For this purpose assume that we are in the situation (1.2.20). Consider the normalization maps $\tilde{\Delta} \longrightarrow \Delta$ and $\tilde{\Sigma} \longrightarrow \Sigma$. By the universal property of the normalization, the composed map

$\tilde{\Delta} \longrightarrow \tilde{\Sigma} \longrightarrow \Sigma$ can be lifted to a map $\tilde{\Delta} \longrightarrow \Delta$. These maps fit into a diagram with exact rows and colomns:

(1.2.21) **Diagram :**

$$
\begin{tikzcd}
\mathcal{O}_\Sigma \arrow[r, hook] \arrow[d, hook] & \mathcal{O}_\Delta \arrow[r, two heads] \arrow[d, hook] & \mathcal{O}_\Delta/\mathcal{O}_\Sigma \arrow[d] \\
\mathcal{O}_{\tilde{\Sigma}} \arrow[r, hook] \arrow[d, two heads] & \mathcal{O}_{\tilde{\Delta}} \arrow[r, two heads] \arrow[d, two heads] & \mathcal{O}_{\tilde{\Delta}}/\mathcal{O}_{\tilde{\Sigma}} \\
\mathcal{G}_\Sigma \arrow[r] & \mathcal{G}_\Delta &
\end{tikzcd}
$$

From this one deduces by the snake lemma the exact sequence:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_\Delta/\mathcal{O}_\Sigma \longrightarrow \mathcal{O}_{\tilde{\Delta}}/\mathcal{O}_{\tilde{\Sigma}} \longrightarrow \mathcal{G}_\Delta/\mathcal{G}_\Sigma \longrightarrow 0$$

where $\mathcal{K} := \ker(\mathcal{G}_\Sigma \longrightarrow \mathcal{G}_\Delta) \approx \mathcal{O}_\Delta \cap \mathcal{O}_{\tilde{\Sigma}}/\mathcal{O}_\Sigma$.

(1.2.22) **Lemma :** $\mathcal{H}^0_{\{p\}}(\mathcal{G}_\Sigma) \approx \mathcal{K}$

proof : As $\mathcal{K} \subseteq \mathcal{O}_\Delta/\mathcal{O}_\Sigma$ we see that $\mathcal{K}$ is torsion, so $\mathcal{K} \subseteq \mathcal{H}^0_{\{p\}}(\mathcal{G}_\Sigma)$.
Now let $\rho : \Delta \longrightarrow \Sigma$ the normalization map. Then $\mathcal{H}^0_{\{p\}}(\rho_*\mathcal{G}_\Delta)$, because $\mathcal{O}_\Delta \longrightarrow \mathcal{O}_{\tilde{\Delta}} \longrightarrow \mathcal{G}_\Delta$ is split by normality of Δ (see (1.2.18)). From this one gets exact sequences:

$$0 \longrightarrow \mathcal{H}^0_{\{p\}}(\mathcal{K}) \longrightarrow \mathcal{H}^0_{\{p\}}(\mathcal{G}_\Sigma) \longrightarrow \mathcal{H}^0_{\{p\}}(\mathcal{G}_\Sigma/\mathcal{K}) \longrightarrow$$

and $0 \longrightarrow \mathcal{H}^0_{\{p\}}(\mathcal{G}_\Sigma/\mathcal{K}) \longrightarrow \mathcal{H}^0_{\{p\}}(\rho_*\mathcal{G}_\Delta) \longrightarrow \ldots$

So $\mathcal{K} = \mathcal{H}^0_{\{p\}}(\mathcal{K}) \approx \mathcal{H}^0_{\{p\}}(\mathcal{G}_\Sigma)$. ∎

(1.2.23) **Definition :** Let (C,p) be a reduced curve germ and let $\tilde{C} \xrightarrow{n} C$ be the normalization map. The **delta-invariant** $\delta(C,p)$ is the number

$$\delta(C,p) := \dim_{\mathbb{C}}(n_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C)_p \quad (= \dim_{\mathbb{C}}(\mathcal{O}_{\tilde{C}}/\mathcal{O}_C)\ )$$

This δ-invariant of a curve is a simple and very important invariant. It is also called the "virtual number of double points" (see also § 2.1). For a general reduced curve C we put:

$$\delta(C) = \sum_{p \in C} \delta(C,p)$$

(1.2.24) Corollary : Let (X,Σ,p) be a WNCM-surface germ. Let $\tilde{X} \xrightarrow{n} X$ be as usual the normalization map and $\tilde{\Sigma}=n^{-1}(\Sigma)$. Then:

$$\delta(\tilde{\Sigma}) \geq \delta(\Sigma)$$

This is because for such a surface one has by (1.2.20) $\mathscr{H}^0_{\{p\}}(\mathscr{G})=0$, and so by (1.2.22) $\mathscr{K} = 0$. This just means that the map $\mathcal{O}_\Delta/\mathcal{O}_\Sigma \longrightarrow \mathcal{O}_{\tilde{\Delta}}/\mathcal{O}_{\tilde{\Sigma}}$ is injective. The dimension of the first space is $\delta(\Sigma)$, of the second it is $\delta(\tilde{\Sigma})$.

(1.2.25) Example : We will give a class of examples of WNCM-surfaces. In general one can do the following: take a CM space $\tilde{X}$, a codimension 1 CM subspace $\tilde{\Sigma}$ on which a group G acts and let $\tilde{\Sigma} \longrightarrow \Sigma := \tilde{\Sigma}/G$ be the quotient map. Then the push-out X will be a WNCM-space ($\mathcal{O}_\Sigma \longrightarrow \mathcal{O}_{\tilde{\Sigma}}$ is split by the group action).

We consider here the most simple case: $\tilde{X} = \mathbb{C}^2$, $\tilde{\Sigma}$ a curve in $\tilde{X}$ with equation $f(x,y)=0$, and $G = \mathbb{Z}/2$. We may assume that G acts on $\mathbb{C}^2$, and as a linear automorphism. Hence we can distinguish four cases:

A. $x,y \longrightarrow x,-y$, f invariant: $\quad f = F(x,y^2)$

B. $x,y \longrightarrow x,-y$, f anti-invariant: $\quad f = y.F(x,y^2)$

C. $x,y \longrightarrow -x,-y$, f invariant: $\quad f = F(x^2,x.y,y^2)$

D. $x,y \longrightarrow -x,-y$, f anti-invariant: $\quad f = x.G(x^2,x.y,y^2) + y.H(x^2,x.y,y^2)$

Using (1.2.9) we can write down a set of generators for the ring of the push-out. The result is:

Case A. Generators: $U=x \quad Y=y^2 \quad Z=y.F$

Relation : $Z^2=Y.F(U,Y)^2$

Case B. Generators: $U=x \quad Y=y^2 \quad Z=y.F$

Relation : $Z^2=Y.F(U,Y)^2$

So these cases are really the same; an anti-invariant splits off a factor y, whose image in X does not correspond to a component of the singular locus.

Case C. Generators: $U=x^2$ $Y=y^2$ $Z=x.y$ $S=x.F$ $T=y.F$

Relations : $Z^2=U.Y$ $Z.S=U.T$ $Z.T=Y.S$

$S^2=U.F^2$ $S.T=Z.F^2$ $T^2=Y.F^2$

From the structure of these relations it is clear that we can write every polynomial $P(U,Y,Z,S,T)$ modulo these relations uniquely as $P \equiv a(U,Y) + b(U,Y).Z + c(U,Y).S + d(U,Y).T$. From this it follows that the above equations generate the ideal of relations between the generators. The elements U,Y form a system of parameters; the multiplicity of X is 4 and its Gorenstein type is 3. It is never determinantal (see (1.3.9)).

Case D. Generators: $U=x^2$ $Y=y^2$ $Z=x.y$ $S=f$

Relations : $Z^2 = U.Y$ $S^2 = U.G^2+ 2.Z.G.H + Y.H^2$

Modulo these two relations every polynomial $P(U,Y,Z,T)$ can be written as $P \equiv a(U,Y) + b(U,Y).Z + c(U,Y).S + d(U,Y).S.Z$. Again we find that $\{U, Y\}$ is a system of parameters for the ring and that the two equations generate the ideal of relations. Hence X is a complete intersection of multiplicity 4 in $\mathbb{C}^4$. (The element $\omega = dx\wedge dy/f$ can be considered as a generator of the dualizing sheaf on X).

(1.2.26) Remark : D. Mond [Mo] has given a list of mapping from $\mathbb{C}^2$ to $\mathbb{C}^3$, which are *simple* with respect to left-right coordinate transformations. Finite determinacy for such maps comes down to weak normality of the image. All simple germs happen to be of corank one, so can be written as

$$(x,y) \longrightarrow (x,p(x,y),q(x,y))$$

For those of multiplicity 2 one can take $p(x,y)=y^2,q(x,y)=y.f(x,y^2)$ so these are precisely the above singularities of type A/B. Such a singularity happens to be simple exactly when the curve $f(x,y)=0$ is simple in the ordinary sense, and has $\mathbb{Z}/2$-action (see [Arn 2]). Besides these there is one other series of simple mappings in his list and is called H_k: $(x,y) \longrightarrow (x,x.y + y^{3k-1},y^3)$.

We will discuss this singularity and related families in higher codimension in (2.3.8).

(1.2.27) Conjecture : An AWN-surface germ is algebraic.

By this we mean that an analytic germ $(X,p) \subset (\mathbb{C}^n,0)$, with $X-\{p\}$ weakly normal, can be defined by *polynomials* after an appropriate analytic coordinate transformation. This is known for general spaces with an isolated singular point, by theorems of Artin, Tougeron, and Bochnak (for references see [Pe]). From a theorem of Pellikaan (see [Pe]) it follows in particular that the conjecture is true for surfaces in $\mathbb{C}^3$. Theorem (1.2.20) states that the study of WNCM-surface germs X is equivalent to the the study of diagrams

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{i} & \tilde{X} \\ \downarrow & & \\ \Sigma & & \end{array}$$

with certain properties. This leads to a strategy to prove the stated conjecture:

Step 1. $\tilde{X},\tilde{\Sigma}$ and Σ being a priori analytic spaces with isolated singularities, are in fact analytically isomorphic to germs that can be described by polynomial ideals, by the above mentioned theorem.

Step 2. The maps i and n are analytic maps between spaces with isolated singularities. The following should be true: Any pair of maps (i',n') sufficiently near to (i,n) is conjugate to (i,n) by an automorphism of $\tilde{X},\tilde{\Sigma},\Sigma$ (finite determinacy).

Step 3. Artin's approximation theorem should then garantee the existence of an (i',n') conjugate to (i,n) in the Hensel functions, and hopefully even in the polynomials.

Step 4. The push-out is then analytically isomorphic to a space defined by a polynomial ideal.

Maybe this scheme can be made to work. But then we are confronted with the more general and natural

Question : Is every analytic space (X,p) with the property that (X,q) is algebraic for $q \neq p$, algebraic at p ?

§ 1.3 The Partition Singularities

We introduce a particularly simple class of weakly normal Cohen-Macaulay surface singularities, which we call partition singularities. These spaces can be considered as generalizations of the singularities A_∞ and D_∞ in higher embedding dimension. The partition singularities play a fundamental role in the theory of WNCM-surface singularities, as they appear as unavoidable singularities on an improvement (§ 1.4).

(1.3.1) Definition of a partition singularity:

Let $\pi = (\alpha(1),\alpha(2),\ldots,\alpha(k))$, $\alpha(i) \in \mathbb{N}$, be a partition of the number $\alpha := \sum \alpha(i)$ into k parts.

Let $\tilde{X}_i$, $i=1,\ldots,k$, be a copy of $\mathbb{C}^2$, equipped with coordinates (u_i,v_i), and put $\tilde{\Sigma}_i = \{v_i = 0\} \subset \tilde{X}_i$.

Let Σ be a copy of $\mathbb{C}$, with coordinate u. Consider the mapping $\tilde{\Sigma} := \coprod \tilde{\Sigma}_i \longrightarrow \Sigma$, given by $u = (u_1^{\alpha(1)}, u_2^{\alpha(2)},\ldots,u_k^{\alpha(k)})$.

Put $\tilde{X} := \coprod \tilde{X}_i$. We have an inclusion map $\tilde{\Sigma} \longrightarrow \tilde{X}$ and a finite map $\tilde{\Sigma} \longrightarrow \Sigma$, so we can form the push-out as in § 1.2 (1.2.6) :

$$\begin{array}{ccc} \tilde{\Sigma} & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ \Sigma & \longrightarrow & X \end{array}$$

The space $X =: X_\pi$ we call the partition singularity of type π or the π-partition singularity. Further, we also call every germ (X,Σ,p) locally analytic isomorphic to X_π a partition singularity of that type.

(1.3.2) Definition: Let X_π be a partition singularity, with $\pi = (\alpha(1), \alpha(2), \ldots, \alpha(k))$. Then the image the line $u_i = 0$ on X_π will be called *the special line* L_i. So on X_π there are k special lines L_i, on every irreducible component of X_π one. The special lines are transverse to Σ.

(1.3.3) Lemma : Let (X,Σ,p) WNCM-surface germ. Then X is a partition singularity iff $\tilde{X}$ and $\tilde{\Sigma}$ are *smooth*.

proof : As X is CM and $\tilde{\Sigma}$ is smooth, it follows from (1.2.24) that Σ is also smooth. Choosing an appropriate neighbourhood of p in Σ we may assume that the map $\tilde{\Sigma} \longrightarrow \Sigma$ branches at most over p. As $\tilde{X}$ is smooth, we can find coordinates bringing this map into normal form, as in (1.3.1). ∎

(1.3.4) Lemma : The ring of functions X_π is generated by

$$x := (u_1^{\alpha(1)}, u_2^{\alpha(2)}, \ldots, u_k^{\alpha(k)})$$

$$y_{i,j} := (0,\ldots,0,\ u_i{}^j.v_i,0,\ldots,0) \qquad \begin{array}{l} i=1,2,\ldots,k \\ j=0,1,\ldots,\alpha(i)-1 \end{array}$$

So in total we have $\alpha + 1$ generators, giving a (minimal) embedding of $X_\pi \longrightarrow \mathbb{C}^{\alpha+1}$.

proof : One can use (1.2.9) to find that the x and $y_{i,j}$ generate the local ring of X_π . The minimality is evident from the explicit form of the generators. ∎

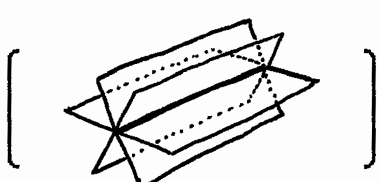
(1.3.5) Examples :

$\pi = (1)$ $\quad X_{(1)} \approx \mathbb{C}^2$, so this not really a singularity. This is a reason why partition singularities with a '1' appearing in π , behave slightly different from the other partition singularities.

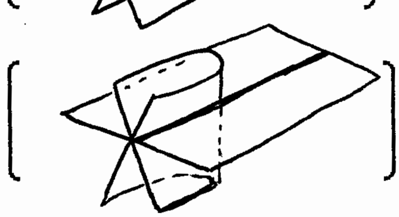
$\pi = (1,1)$ $\quad X_\pi \approx A_\infty$

$\pi = (2)$ $\quad X \approx D_\infty$

$\pi = (1,1,1)$ $\quad X_\pi =$ WN



$\pi = (2,1)$ $\quad X_\pi =$ WN



$\pi = (3)$ $\quad$ No picture available.

(Here WN means: 'weak normalization of'.)

(1.3.6) Lemma : If $n \geq 2$ the ideal of $X_{(n)}$ is generated by the 2×2-minors of the matrix:

$$M = \begin{bmatrix} y_0 & y_1 & \cdots & y_{n-2} & y_{n-1} \\ y_1 & y_2 & \cdots & y_{n-1} & x\cdot y_0 \end{bmatrix}$$

Here $y_i := y_{1,i}$ in the notation of (1.3.4).

proof : Let Y be the variety defined by the 2×2-minors of the matrix M. It is clear that $X_{(n)} \subseteq Y$. As $X_{(n)}$ and Y are both CM and are generically equal, it follows that $X_{(n)} = Y$ □
Alternatively, the elements x and y_0 form a system of parameters for the coordinate ring of Y: every element P in this ring has a unique representation as $P = \sum a_i(x,y_0)\cdot y_i + b(x,y_0)$. Pulling this back to the normalization of $X_{(n)}$, it is seen that $P = 0$ is equivalent to $a_i = b = 0$ ■

(1.3.7) Corollary : $\mathrm{Mult}(X_\pi) = \alpha$, $\mathrm{type}(X_\pi) = \alpha-1$ $(\pi \neq (1))$

As the general X_π is obtained by joining $X_{(\alpha(i))}$ along their singular line, the ideal of X_π is generated by the following system of

(1.3.8) Equations :

A) $$\mathrm{rank}\begin{bmatrix} y_{i,0} & y_{i,1} & \cdots & y_{i,\alpha(i)-2} & y_{i,\alpha(i)-1} \\ y_{i,1} & y_{i,2} & \cdots & y_{i,\alpha(i)-1} & x\cdot y_{i,0} \end{bmatrix} \leq 1$$

B) $$y_{i,p}\cdot y_{j,q} = 0 \qquad \begin{array}{c} p=0,1,\ldots,\alpha(i)-1 \\ q=0,1,\ldots,\alpha(j)-1 \\ i,j=1,2,\ldots,k\ ;\ i\neq j \end{array}$$

So in total we have $\alpha.(\alpha-1)/2$ generators for the ideal. This is equal to the numbers of 2×2-minors of a $2\times\alpha$ matrix.
Is X_π "determinantal" ?

(1.3.9) Definition : A singularity is called *determinantal* if its equations can be written as the $m\times m$-minors of an $p\times q$-matrix and if it has the same codimension as the *generic determinantal singularity* $(=(p-m+1).(q-m+1))$, see

[Lak], [Scha 1]). So a determinantal singularity can be seen as the pull-back of the generic determinantal singularity under a transversal map. They are Cohen-Macaulay and their projective resolution is known. For example, complete intersections are determinantal (m=p=1), but also every CM space of codimension two is (m=p=q-1, "Hilbert's theorem", see [Scha 1]).

Define

$$M_n(\underline{Y}) = \begin{bmatrix} y_0 & y_1 & \cdots & y_{n-2} & y_{n-1} \\ y_1 & y_2 & \cdots & y_{n-1} & x\cdot y_0 \end{bmatrix} \qquad n > 1$$

$$M_1(\underline{Y}) = \begin{bmatrix} y_0 \\ 0 \end{bmatrix}$$

where $\underline{Y}$ denotes a sequence of variables $y_0, y_1, \ldots, y_{n-1}$.
Let $A_i \in \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGl}_2(\mathbb{C})$, i=1,2,...,k and write

$$A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$$

Consider the following $\alpha\times 2$ matrix $M_\pi = M_{\pi,A}(\underline{Y})$:

$$M_{\pi,A}(\underline{Y}) = \begin{bmatrix} A_1\cdot M_{\alpha(1)}(Y^{(1)}) & A_2\cdot M_{\alpha(2)}(Y^{(2)}) & \cdots & A_k\cdot M_{\alpha(k)}(Y^{(k)}) \end{bmatrix}$$

where $Y^{(i)}$ denotes the set of variables $Y_{i,j}$, $j=0,1,\ldots\alpha(i)-1$.

(1.3.10) **Proposition :** In a neighbourhood of $0 \in \mathbb{C}^{\alpha+1}$, the ideal of X_π is generated by the 2×2-minors of the matrix M_π if and only if the points $p_i = (a_i:c_i) = A_i\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{P}^1$, $i=0,1,\ldots,k$ are all distinct.

Proof : There are two kinds of 2×2 minors:

a) minors involving one $M_{\alpha(i)}(Y^{(i)})$

b) minors involving two M_α 's.

The minors of type a) give just part A) of the equations (1.3.8), because we assumed the A_i to be non-singular. For the minors of type b) we have to study the mixed minors of, say

$$\left[\; A_1 \cdot M_{\alpha(1)}(Y^{(1)}) \quad A_2 \cdot M_{\alpha(2)}(Y^{(2)}) \;\right]$$

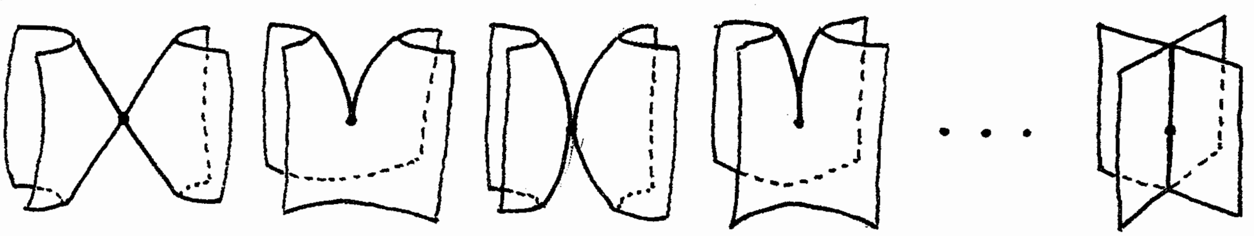
After performing the linear transformation A_2^{-1}, we may assume that

$$A_1 = \begin{bmatrix} a & d \\ c & d \end{bmatrix}, \text{ with } c \neq 0 \text{ , } A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

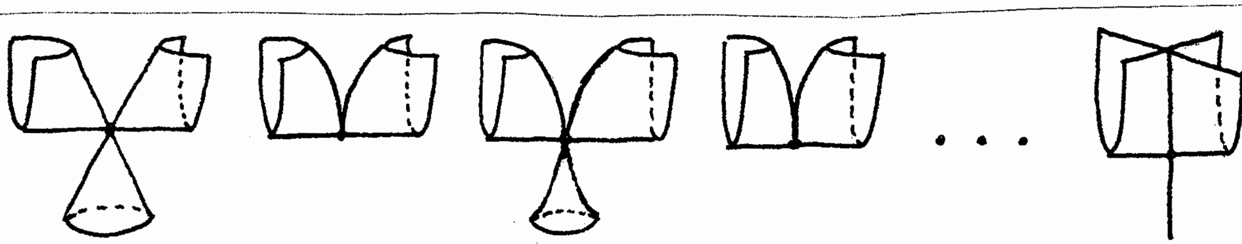
All mixed minors have the form $\sum W_{i,j}^{k,l} \cdot y_i^{(1)} \cdot y_j^{(2)}$, with $W_{i,j}^{k,l} \in \mathbb{C}\{x\}$, depending linearly on a,b,c,d. So these minors generate the same ideal iff $\det(W_{i,j}^{k,l})$ is a unit in $\mathbb{C}\{x\}$. For $a = b = d = 0$ this determinant is seen to be equal to $c^{\alpha(1).\alpha(2)}$, so a unit in $\mathbb{C}\{x\}$. For a,b,d small this remains true and by homogeneity the result follows. ∎

The importance of representing X_π as a determinantal singularity is that it enables us to construct flat deformations of a very special kind, by perturbing the entries of the matrix (see [Scha 2]).

Recall that A_∞ and D_∞ give rise to the series A_k and D_k .

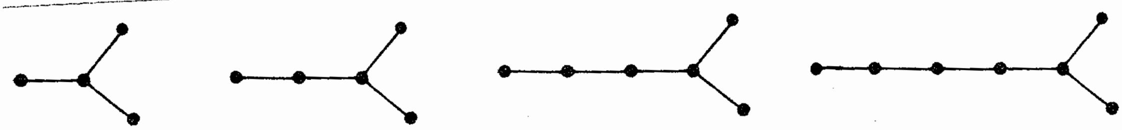


A_1 A_2 A_3 A_4 A_∞

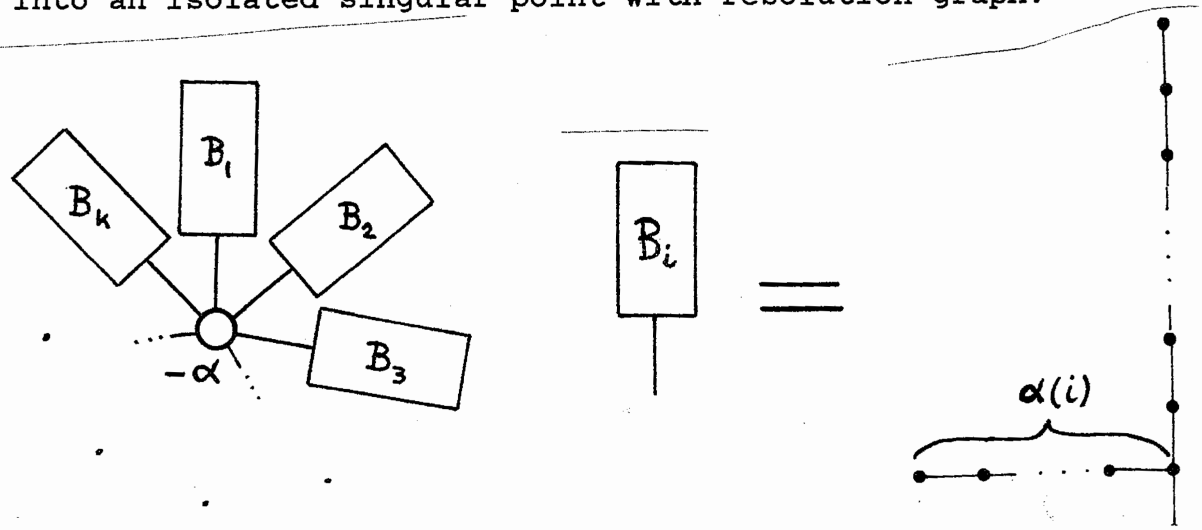


D_4 D_5 D_6 D_7 D_∞

The corresponding resolution graphs (§ 1.4) look like:



We are going to exhibit a deformation of X_π , $\pi=(\alpha(1),\alpha(2),\ldots\alpha(k))$ into an isolated singular point with resolution graph:



There is is a central $\mathbb{P}^1$, with self-intersection $-\alpha$, and k branches coming out of this. Each branch splits up into two "legs": one with length $\alpha(i)$, the other with an arbitrary length.

We first describe a modification that can be performed on every $2\times\alpha$-determinantal singularity. I call it the "Tjurina-modification" because to my knowledge it first (implicitly) appeared in [Tj] (in any case it sounds good).
Let $\mathcal{F}$ and $\mathcal{G}$ be free $\mathcal{O} = \mathcal{O}_{\mathbb{C}^{\alpha+1}}$ - modules of rank α and 2 respectively. Consider a map $\mathcal{F} \xrightarrow{\Phi} \mathcal{G}$. This gives rise to a map $\Lambda^2\mathcal{F} \otimes \Lambda^2\mathcal{G}^* \longrightarrow \mathcal{O}$ with as cokernel the structure sheaf of the space X defined by the 2×2-minors of $[\Phi]$. When we transpose the map Φ we get a $\mathcal{G}^* \longrightarrow \mathcal{F}^*$ and tensoring with $\mathcal{O}_X$ we get an

exact sequence

$$0 \longrightarrow \mathcal{K}_X \longrightarrow \mathcal{G}^* \otimes \mathcal{O}_X \longrightarrow \mathcal{F}^* \otimes \mathcal{O}_X$$

where $\mathcal{K}_X$ is a certain torsion free module on X. When X is determinantal, $\mathcal{K}_X$ is of rank 1. Going to the associated spaces, we get $\mathbb{P}(\mathcal{K}_X) \subset \check{\mathbb{P}}^1_X = \check{\mathbb{P}}^1 \times X$, $(\mathbb{P}^1 = \mathbb{P}(\mathcal{G}/\mathfrak{m}.\mathcal{G}))$.
At a generic point of X the map $\mathbb{P}(\mathcal{K}_X) \longrightarrow X$ is 1:1. The closure of these components in $\check{\mathbb{P}}^1_X$ we call $\bar{X}$ and the resulting map $\bar{X} \longrightarrow X$ we call the "Tjurina-modification". The space $\check{\mathbb{P}}^1 = \mathbb{P}((\mathcal{G}/\mathfrak{m}.\mathcal{G})^*) \subset \mathbb{P}(\mathcal{K}_X)$ we call the " central $\mathbb{P}^1$ " of the modification. Note that there is a confusing natural isomorphism $\mathbb{P}^1 \longrightarrow \check{\mathbb{P}}^1$.

(1.3.11) Example : Let $[\Phi] = \begin{bmatrix} f_1 & f_2 & \dots & f_n \\ g_1 & g_2 & \dots & g_n \end{bmatrix}$

What we described as $\mathbb{P}(\mathcal{K}_X)$ is a pedantic way of talking about the system of equations $s.f_i = t.g_i$, $i=1,\dots,n$; $(s:t) \in \check{\mathbb{P}}^1$.

Take for instance:

$$[\Phi] = \begin{bmatrix} y_0 & y_1 & \dots & y_{n-2} & y_{n-1} \\ y_1 & y_2 & \dots & y_{n-1} & x.y_0 \end{bmatrix}$$

Then we get

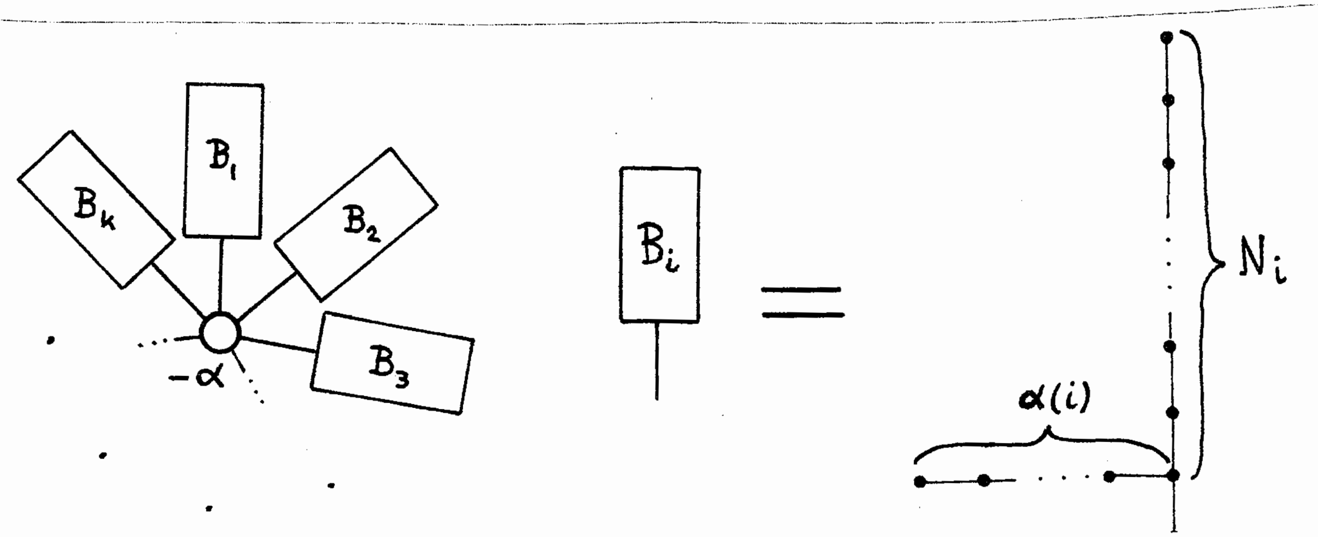
$$\begin{aligned} s.y_0 &= t.y_1 & \qquad s.y_{n-1} &= t.x.y_0 \\ s.y_1 &= t.y_2 & & \\ &\ \vdots & i&=0,1,\dots,n-2 \\ s.y_i &= t.y_{i+1} & & \end{aligned}$$

On the chart $t \neq 0$ we can write $y_i = s^i.y_0$ $i=0,1,\dots,n-1$ and so $(s - x).y_0 = 0$. Hence $\mathbb{P}(\mathcal{K}_X)$ has two components:

1) $y_i = 0 \approx \check{\mathbb{P}}^1 \times \Sigma$.
2) A smooth component $\tilde{X}$ mapping to $X_{(n)}$ as the normalization.

When we take the full matrix M_π we get k smooth components instead of one. They intersect $\check{\mathbb{P}}^1$, the fibre over 0 of the Tjurina-modification , in the points q_i corresponding to the p_i of (1.3.10) under the *natural isomorphism* $\mathbb{P}^1 \longrightarrow \check{\mathbb{P}}^1$.

(1.3.12) **Theorem :** Let X_π be a partition singularity of type $\pi = (\alpha(1),\alpha(2),\dots,\alpha(k))$. For every choice of points $q_i \in \mathbb{P}^1$, and numbers $N_i \geq 0$, $i = 1,2,\dots,k$, there is a flat deformation $F : \mathcal{X} \longrightarrow \mathbb{C}^k$ of X_π such that the general fibre $F^{-1}(\underline{\lambda})$ has a single isolated singular point with a resolution graph of the following form:



The central $\mathbb{P}^1$ has self-intersection $-\alpha$ and the points q_i can be identified with the intersection points of this $\mathbb{P}^1$ with the k other $\mathbb{P}^1$'s.

Proof : Determine numbers ε_i and r_i by the relations

$$N_i = \alpha(i).r_i - \varepsilon_i \qquad 0 \leq \varepsilon_i \leq \alpha(i)-1.$$

Choose matrices $A_i \in \mathrm{Aut}(\mathbb{P}^1)$ such that the points $p_i = A_i.\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ correspond to the points q_i.

Define the following matrices:

$$M_n(\underline{Y},\lambda,\varepsilon,r) = \begin{bmatrix} y_0 & y_1 & \dots & y_\varepsilon + \lambda.x^r & \dots & y_{n-2} & y_{n-1} \\ y_1 & y_2 & & \dots & & y_{n-1} & x.y_0 \end{bmatrix} \qquad n\geq 2$$

$$M_1(\underline{Y},\lambda,\varepsilon,r) = \begin{bmatrix} y_0 \\ \lambda.x^r \end{bmatrix}$$

$$M_i = A_i.M_{\alpha(i)}(\underline{Y}^{(i)},\lambda_i,\varepsilon_i,r_i) , \qquad i = 1,2,\dots,k.$$

$$M(\underline{Y},\underline{\lambda}) = \begin{bmatrix} M_1, & M_2, & \dots, & M_k \end{bmatrix}$$

Let $\mathcal{X} := \{ (\underline{Y},\underline{\lambda}) \in \mathbb{C}^{\alpha+1} \times \mathbb{C}^k \mid \text{rank}(M(\underline{Y},\underline{\lambda})) \le 1 \}$ and $F : \mathcal{X} \longrightarrow \mathbb{C}^k$ the evident projection map.

Let $\bar{\mathcal{X}} \longrightarrow \mathcal{X}$ be the Tjurina modification. Around the point q_i we have to analyse the system of equations, coming from the matrix M_i. So let us first study the modification for the matrix

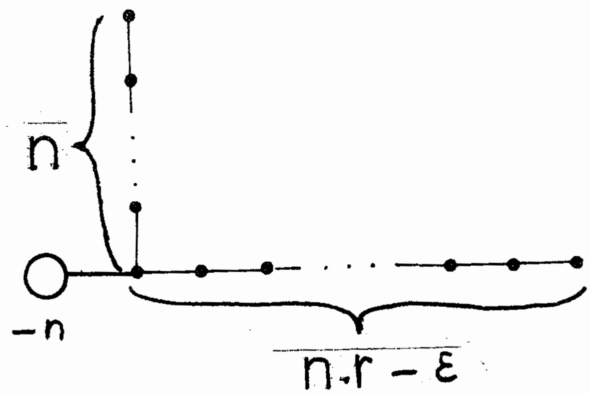
$$\begin{bmatrix} y_0 & y_1 & \dots & y_\varepsilon + \lambda . x^r & \dots & y_{n-2} & y_{n-1} \\ y_1 & y_2 & & \dots & & y_{n-1} & x . y_0 \end{bmatrix}$$

This leads to the system of equations:

$$\begin{aligned} s.y_0 &= t.y_1 \\ s.y_1 &= t.y_2 \\ &\vdots \\ s.(y_\varepsilon + \lambda.x^r) &= t.y_{\varepsilon+1} \\ &\vdots \\ s.y_{n-1} &= t.x.y_0 \end{aligned}$$

In the chart $t = 1$ we have $y_i = s^i.y_0$, $i = 0,1,\dots,\varepsilon$, and from this $s.(s.y_0 + \lambda.x) = x.y_0$. When we introduce a new coordinate $\bar{x} = s^n - x$, this can be written as $\bar{x}.y_0 + \lambda.s^{n-\varepsilon}.(s^n - \bar{x})^r = 0$, or $\bar{x}.y_0 + \lambda.s^{n(r+1)-\varepsilon} + \bar{x}.g(\bar{x},s) = 0$, for a certain g. With $\bar{y} = y_0 + g(\bar{x},s)$ we get $\bar{x}.\bar{y} + \lambda.s^{n(r+1)-\varepsilon} = 0$.

Hence we see that the Tjurina modification belonging to the above matrix has an A_m- singularity at $(1:0)\times\underline{0}\times\lambda$ for $\lambda \neq 0$ and $m=n(r+1)-\varepsilon-1$. When we resolve this A_m- singulariy we introduce a chain of m $\mathbb{P}^1$'s. But then we still have to determine where the strict transform of the central $\mathbb{P}^1$ intersects this chain. It is easy to verify that the curve $t \longrightarrow (t^a, t^{m+1-a}, t)$ on the germ $x.y = z^{m+1}$, $a = 1,2,\dots,m$ has a strict transform cutting the a-th curve of the chain of $\mathbb{P}^1$'s on the minimal resolution. In the coordinates x,y_0,s the curve corresponding to the central $\mathbb{P}^1$ is given by $t \longrightarrow (0,0,t)$. Performing the coordinate changes it becomes the curve $t \longrightarrow (\bar{x},\bar{y},s)=(t^n,t^{n.r-\varepsilon},t)$. So the central $\mathbb{P}^1$ intersects precisely n-th curve of the chain and so the resolution looks like:



The point that needs some clarification is the fact that the self-intersection of the central $\mathbb{P}^1$ is really -n. This can be checked directly, by looking at the multiplicities of the function X of (1.3.4) on all exceptional divisors.
The theorem follows from this, because around each point q_i we have essentially a situation as in the above computation.(Only the cases in which '1' appears in the partition π are slightly different.) ∎

(1.3.13) **Remark :** The isolated singularities that appear in the generic fibre of the family in (1.3.12) are certain special rational determinantal singularities.
The special case corresponding to $\pi = (1,1,\ldots,1)$ gives rise to the ones with reduced fundamental cycle (see § 3.4) and can be found in [Wah 1].
The cases corresponding to $\pi = (1,1,1)$, $(2,1)$, (3) were given by Tjurina in [Tj] and give rise to certain rational triple points (see (4.1.24)).

§ 1.4 Improvements

The notion of improvement was introduced by N.Shepherd Barron in [Sh], as a tool for the study of a certain class of non-isolated surface singularities (the degenerate cusps, see also § 4.3). The idea is as follows: Normal surface singularities (X,p) are usually studied using a resolution $Y \xrightarrow{\pi} X$. Due to the fact that $\pi_*\mathcal{O}_Y = \mathcal{O}_X$, all information about X is contained in the space Y. However, when we resolve a non-isolated surface singularity (X,Σ,p), we can not have $\pi_*\mathcal{O}_Y = \mathcal{O}_X$, because the resolving map changes the space X in codimension 1. In particular we lose information about the singular locus Σ when we only look at the space Y. To remedy this, we do not resolve X, but only allow modifications in codimension two, and look for the *best possible* space Y we can get that way. Such a space Y is allowed to have certain very special singularities and will be called an improvement of X.

(1.4.1) Resolutions.

We recall some facts about resolutions.

Definition : Let X be a generically reduced space.
A space Y together with a map $Y \xrightarrow{\pi} X$ is called a *resolution* of X if and only if the following conditions hold:

1) π is a proper map.
2) $Y - \pi^{-1}(\Sigma) \longrightarrow X - \Sigma$ is an isomorphism.(here Σ is the singular locus of X and $\pi^{-1}(\Sigma)$ is required to be nowhere dense in Y).
3) Y is smooth.

(A shorter way to formulate 2) is to say that π is bimeromorphic.)
We refer to π as the *resolving map* or *contraction map* and to Y as the *resolving space* or even the *resolution*. Condition 2) means that we modify our space X only inside Σ to obtain a smooth model for X. We call the set $\pi^{-1}(\Sigma)$ the exceptional set of π (or sometimes of Y, if no confusion can arise).
It is a theorem of Hironaka that every space has a resolution ([Hi]).

For surfaces there are several ways to construct resolutions. First normalize X to get a surface $\tilde{X}$ with isolated singularities. Blowing up these points gives a space X_1 which might be non-normal. Repeating normalization and blowing up points lead in a *finite number of steps* to a space which is smooth and hence results in a resolution of X. This a theorem of Zariski (see [Za 1]). Another way to obtain a resolution of a surface is by first projecting X generically to a plane H . The map $X \longrightarrow H$ branches over a curve $B \subset H$. By repeated blowing up points the branch curve is transformed into a normal crossing divisor $\bar{B}$ in the blown up plane $\bar{H}$. Pulling back X we get a space $\bar{X} \longrightarrow \bar{H}$, whose normalization has only *cyclic quotient singularities* above the intersection points of $\bar{B}$, and the resolution of these singularities is easy.(Hirzebruch-Jung method, see [B-P-V], [Lau 1]). We refer to [Za 2] and [Ab] for general information about the history of the resolution process.

We now take a closer look at the resolution $Y \xrightarrow{\pi} X$ of normal surface germ (X,p). The exceptional divisor $E := \pi^{-1}(p)$ is a compact curve and is the 'maximal' compact subset of Y. There exists a unique *minimal* resolution $Y_0 \longrightarrow X$ such that every other resolution factorizes over this one. It is characterized by the property that Y_0 does not contain any *exceptional curve of the first kind*, that is, a smooth $\mathbb{P}^1$ with self-intersection -1.(see [Lau 1]). A resolution $Y \xrightarrow{\pi} X$ is called a *good resolution* if the curve E is a normal crossing divisor on Y, and all irreducible components E_i of E are smooth. Again there is a *minimal good* resolution with the property that it is dominated by all good resolutions. This minimal good resolution is obtained by starting with the minimal resolution and then resolving E embeddedly in Y in the minimal way to a normal crossing divisor. For such a good resolution it is customary to encode the discrete data of the resolution into the *resolution graph*. It is a graph with one vertex for each irreducible component E_i and one edge between two vertices for every point of intersection. Each vertex has two numbers attached to it:

1) $g(E_i)$, the genus of the curve E_i .

2) E_i^2 , the self-intersection number of E_i .

Usually genus 0 and self-intersection -2 are suppressed from the notation.

example : The simple surface singularities A_n $(n\geq 1)$, D_n $(n\geq 4)$, E_n $(n=6,7,8)$ have as resolution graph (of the minimal = minimal good resolution) just the Dynkin diagram of their name (see [S1]).

(1.4.2) Improvements.

We propose the following general definition of an improvement, as it arose from discussions with J. Stevens.

Definition : Let X be a generically reduced space.
A space Y together with a map $Y \xrightarrow{\pi} X$ is called an *improvement* if and only if the following conditions hold:

1) π is a proper map.
2) There exists $M \subset \Sigma := \mathrm{Sing}(X)$,with codim $M \geq 2$ such that $Y - \pi^{-1}(M) \longrightarrow X - M$ is an isomorphism (and $\pi^{-1}(M)$ is nowhere dense in Y).
3) $\Delta := \mathrm{Sing}(Y)$ is smooth of codimension 1 (or empty) and equal to the strict transform of Σ under π.
4) Let $\tilde{Y} \xrightarrow{n} Y$ be the normalization map and put $\tilde{\Delta} = n^{-1}(\Delta)$. Then $\tilde{Y}$ and $\tilde{\Delta}$ are smooth and Y is CM.

(Σ, Δ and $\tilde{\Delta}$ are understood with their reduced structure.)

The idea is to modify X only in codimension 2 and ask for the best possible properties for Y. Note that when $\mathrm{codim}(\Sigma) \geq 2$ we can take $M = \Sigma$ and recover the ordinary notion of resolution.
In general however, it is totally unclear under what conditions improvements in the above sense exist (c.f.(2.5.12)). Probably it is better to weaken 3) and 4) by allowing certain mild singularities on Δ and $\tilde{\Delta}$ when $\dim(X) \geq 3$. As we are interested primarily in surfaces, we will not discuss the general problem, although the 3-dimensional case is of importance for the deformation theory of surfaces over a curve.

(1.4.3) **Proposition :** Let (X,Σ,p) be an AWN-surface germ.

Then there exists an improvement $Y \xrightarrow{\pi} X$.
The space Y has only partition singularities.

proof : We construct an improvement for X in 5 steps. Remember that by convention $\dim(\Sigma) = 1$.

STEP 1: Normalize X and get $\tilde{X} \xrightarrow{n} X$ and put $\tilde{\Sigma} = n^{-1}(\Sigma)$. The induced map $\tilde{\Sigma} \longrightarrow \Sigma$ is a finite covering, branched above $p \in \Sigma \subset X$.

STEP 2: Take an embedded resolution $\tilde{Y} \xrightarrow{\rho} \tilde{X}$ of the curve $\tilde{\Sigma}$ in $\tilde{X}$. Let $\tilde{\Delta}$ be the strict transform of $\tilde{\Sigma}$ on $\tilde{Y}$:

$$\tilde{\Delta} = \overline{\rho^{-1}(\tilde{\Sigma} - n^{-1}(p))}$$

We may assume that $\tilde{\Delta}$ is transverse to the exceptional set $\tilde{E} = \rho^{-1}(n^{-1}(p))$.

STEP 3: We have a composed map $\tilde{\Delta} \longrightarrow \tilde{\Sigma} \longrightarrow \Sigma$, which finite. As $\tilde{\Delta}$ is smooth, this map factorizes over the normalization mapping $\Delta \longrightarrow \Sigma$ of Σ, and we have a diagram:

$$\begin{tikzcd} \tilde{\Delta} \arrow[r] \arrow[d, "\rho"'] & \Delta \arrow[d] \\ \tilde{\Sigma} \arrow[r, "n"] & \Sigma \end{tikzcd}$$

STEP 4: We have maps $\tilde{\Delta} \longrightarrow \Delta$ and $\tilde{\Delta} \longrightarrow \tilde{Y}$. By (1.2.6) we can form the space Y , obtained from $\tilde{Y}$ by glueing $\tilde{\Delta}$ to Δ. We obtain a push-out diagram as in (1.2.2):

$$\begin{tikzcd} \tilde{\Delta} \arrow[r] \arrow[d] & \tilde{Y} \arrow[d] \\ \Delta \arrow[r] \arrow[ur, phantom, "\lrcorner"] & Y \end{tikzcd}$$

STEP 5: By the universal property of the push-out (1.2.2)

$$\begin{tikzcd}
\tilde{\Delta} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & \tilde{Y} \arrow[d] \arrow[dr] & \\
\Delta \arrow[r] \arrow[dr] & Y \arrow[dr, dotted] & \tilde{X} \arrow[d] \\
& \Sigma \arrow[r] & X
\end{tikzcd}$$

we get the dotted arrow $Y \longrightarrow X$. So we have constructed a space Y together with a map $Y \longrightarrow X$.

CLAIM : $Y \xrightarrow{\pi} X$ is an improvement (in the sense of (1.4.2))
Conditions 1), 3) and 4) are satisfied by construction. Also, by construction Y has only partition singularities. It is clear that $Y - \pi^{-1}(p) \longrightarrow X - \{p\}$ is an analytic homeomorphism. As we supposed $X - \{p\}$ to be weakly normal, it follows that it is an isomorphism. Hence condition 2) and so $Y \xrightarrow{\pi} X$ is an improvement. ∎

(1.4.4) Definition : A surface, like the improvement Y, which only contains partition singularities will be called *weakly smooth* .

(1.4.5) Remark : It is possible to construct an improvement for a general, not necessarily almost weakly normal surface (X,Σ,p) (J. Stevens [Stev 2]). This can be done as follows: put on Σ and $\tilde{\Sigma}$ the conductor structure, as in (1.2.11). One can define a non-reduced strict transform $\tilde{\Delta}$ for an embedded resolution $\tilde{Y} \longrightarrow \tilde{X}$ of the curve $\tilde{\Sigma}_{red}$ and then form the push-out on $\tilde{\Delta} \longrightarrow \tilde{Y}$ and $\tilde{\Delta} \longrightarrow \Sigma$, giving a space Z mapping generically isomorphic to X. Taking the Cohen-Macaulification $c : Y \longrightarrow Z$ of Z then gives an improvement $Y \xrightarrow{\pi} X$. However, it is not known in general what kind of singularities one has to allow on Y. In the case that X has transverse to its singular locus a *simple singularity* a list of normal form can be given (see [Stev 2]).

(1.4.6) Remark : In the case where X has transversally an A_1-singularity J. Kollar has constructed an *embedded* improvement by blowing up points and smooth curves. The main point is that in the ordinary resolution process one has to blow up along centers with highest multiplicity. Because along the double curve one has multiplicity 2, one first reaches a stage in which one has multiplicity at most 2 everywhere. So one can reduce to a hypersurface situation, which turns out to be analysable. This method does not seem to work for the other transversal types, but it generalizes to higher dimension.(see [K-S]).

(1.4.7) Notation : From now on we will stick to following notational convention:

$$\begin{tikzcd} \tilde{\Delta} \arrow[r] \arrow[d] & \tilde{Y} \arrow[r] \arrow[d] & Y \arrow[d, "\pi"'] & \Delta \arrow[l] \arrow[d] \\ \tilde{\Sigma} \arrow[r] & \tilde{X} \arrow[r, "n"] & X & \Sigma \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} \tilde{\Delta} \arrow[r] \arrow[d] & \Delta \arrow[d] \\ \tilde{\Sigma} \arrow[r] & \Sigma \end{tikzcd}$$

Here
- X is a germ (X,Σ,p) with $X-\{p\}$ weakly normal;
- $\tilde{X}$ is the normalization of X;
- $\tilde{Y}$ is a resolution of $\tilde{X}$, as in the proof of (1.4.3);
- Y is an improvement of X;
- Σ is the singular locus $\mathrm{Sing}(X)$ of X;
- $\tilde{\Sigma}$ is the inverse image $n^{-1}(\Sigma)$ of Σ in $\tilde{X}$;
- Δ is the normalization of Σ;
- $\tilde{\Delta}$ is the normalization of $\tilde{\Sigma}$;
- E is the exceptional set $\pi^{-1}(p)$ of $Y \longrightarrow X$;
- $\tilde{E}$ is the exceptional set of $\tilde{Y} \longrightarrow \tilde{X}$;
- S is the set $\Delta \cap E$ of *special points* on Y;
- $\tilde{S}$ is the set $\tilde{\Delta} \cap \tilde{E}$ of special points on $\tilde{Y}$;

Most of the time we suppress the names of the maps between these spaces, as there is only one sensible map. We also give the same name to similar maps between different spaces, like $\tilde{Y} \xrightarrow{n} Y$. The notational advantage is clear.

(1.4.8) Lemma : Let X be a (pure, generically reduced) surface and let $Y \xrightarrow{\pi} X$ be an improvement. Then there exists an exact sequence:

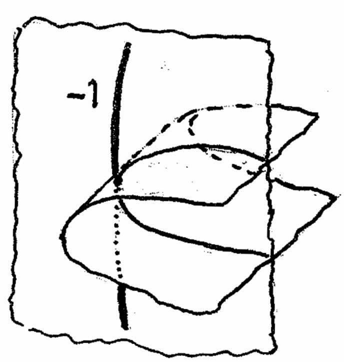
$$0 \longrightarrow \mathcal{H}^0_{\{p\}}(\mathcal{O}_X) \longrightarrow \mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_Y \longrightarrow \mathcal{H}^1_{\{p\}}(\mathcal{O}_X) \longrightarrow 0$$

So in particular $\mathcal{O}_X \approx \pi_*\mathcal{O}_Y$ iff X is CM.

Proof : Because Y is assumed to be CM one can show that $\pi_*\mathcal{O}_Y \approx i_*i^*\mathcal{O}_X$,where $i: X - \{p\} \longrightarrow X$ is the inclusion map. ∎

As already mentioned, a curve C on $\tilde{Y}$ is called an exceptional curve of the first kind if and only if C is a smooth rational curve with self-intersection -1. Such a curve can be blown down without affecting the smoothness of the space $\tilde{Y}$. However, due to the presence of the curve $\tilde{\Delta}$, not all these exceptional curves of the first kind can be blown down on an improvement without affecting its weak smoothness.

(1.4.9) **Example :** Take the singularity of example (1.1.16). It has an improvement that consists of two smooth pieces. One piece is $\mathbb{C}^2$ blown up in one point, the other piece is isomorphic to $\mathbb{C}^2$. This second piece is glued to the first along a curve that is tangent to the exceptional curve of the first piece :



Here the (-1)-curve cannot be blown down without affecting the weak smoothness of the space Y.

(1.4.10) **Definition :** A curve C on $\tilde{Y}$ is said to be a curve of *type k* if and only if it is an exceptional curve of the first kind and $\tilde{\Delta}.C = k$.
An improvement $Y \xrightarrow{\pi} X$ such that $\tilde{Y}$ does not contain curves of type 0 we call *weakly minimal*.

(1.4.11) Proposition : Let X be an AWN surface germ. Then there exists an improvement $Y_0 \longrightarrow X$ with the property that every improvement of X factorizes over Y_0. Y_0 is characterized by the property that $\tilde{Y}_0$ does not contain any curves of type 0 or 1. This improvement is called the <u>minimal improvement.</u>

Proof : Let $Y_1 \longrightarrow X$ be an arbitrary improvement of X. When $\tilde{Y}_1$ contains of type 0, it can be blown down on Y_1 to give another improvement. If $\tilde{Y}_1$ contains a curve C of type 1, so $C^2 = -1$, $C \approx \mathbb{P}^1$, $C.\tilde{\Delta} = 1$, then when we blow down C on $\tilde{Y}_1$ we still have that image of $\tilde{\Delta}$ is smooth, so by the push-out construction we again find an improvement, dominated by Y_1. In a finite number of steps we reach an improvement Y_0 such that $\tilde{Y}_0$ does not contain curves of type 0 or 1. Let $Y_2 \longrightarrow X$ be another improvement, and let $\tilde{Z} \longrightarrow \tilde{X}$ be the minimal resolution of $\tilde{X}$. So both $\tilde{Y}_0 \longrightarrow \tilde{X}$ and $\tilde{Y}_2 \longrightarrow \tilde{X}$ factorize over $\tilde{Z} \longrightarrow \tilde{X}$. Let $\tilde{Y}_0 \xrightarrow{p} \tilde{Z}$ and $\tilde{Y}_2 \xrightarrow{q} \tilde{Z}$ be the resulting maps. Now the first map can be considered as the minimal embedded resolution of the curve $p(\tilde{\Delta}_0) \subset \tilde{Z}$ and the second as another embedded resolution of the same curve. Hence we obtain a map $\tilde{Y}_2 \longrightarrow \tilde{Y}_0$, mapping $\tilde{\Delta}_2$ to $\tilde{\Delta}_0$, so we obtain a map $Y_2 \longrightarrow Y_0$ In other words, Y_0 has the property of the minimal improvement. ■

We also need good improvements.

(1.4.12) Definition : An improvement $Y \xrightarrow{\pi} X$ is called a *good improvement* if and only if the following holds:

1) $\tilde{E}$ is a normal crossing divisor.
2) the $\tilde{E}_i$ are smooth.
3) $\tilde{\Delta}$ intersects the $\tilde{E}_i$ transversally.

Similarly one has:

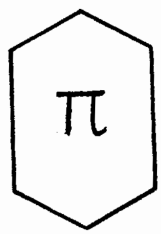
(1.4.13) Proposition : Let X be an AWN surface germ. Then there exists an improvement $Y_1 \longrightarrow X$ with the property that every good improvement of X factorizes over Y_1. Y_1 is characterized by the property that $\tilde{E}_i.(\Sigma_{i\neq j}\ \tilde{E}_j\ +\ \tilde{\Delta}) \geq 3$ for every curve $\tilde{E}_i$ of type 0 or 1.(<u>Minimal good improvement.</u>)

Proof : Similar to the proof of (1.4.12) ∎

(1.4.14) Remark : In Chapter 3 we will need even better than good improvements, called *stable models.*

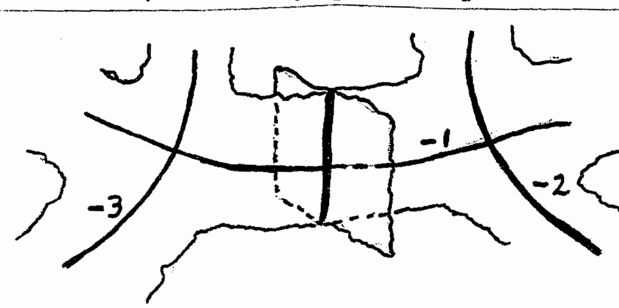
For a good improvement we introduce the *improvement graph* in the same way as we did for a good resolution, but we have to take special care of the partition singularities. We propose the following symbol to denote the occurence of X_π on an improvement:

(1.4.15) Symbol :

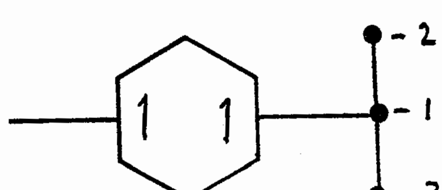


As has been mentioned in (1.3.2), X_π, $\pi = (\alpha(1),\alpha(2),\ldots,\alpha(k))$ consists of k irreducible components and on every component there is a distinguished (class of a) line L_i, transverse to its singular line. So the symbol of X_π is connected to k vertices in the improvement graph.

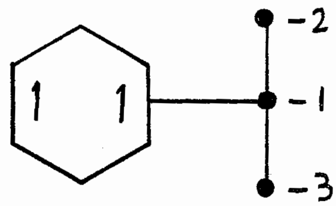
(1.4.16) Example : Take again the example of (1.4.9) The (minimal) good improvement looks like:



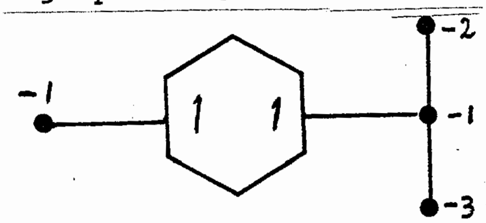
The improvement graph is:



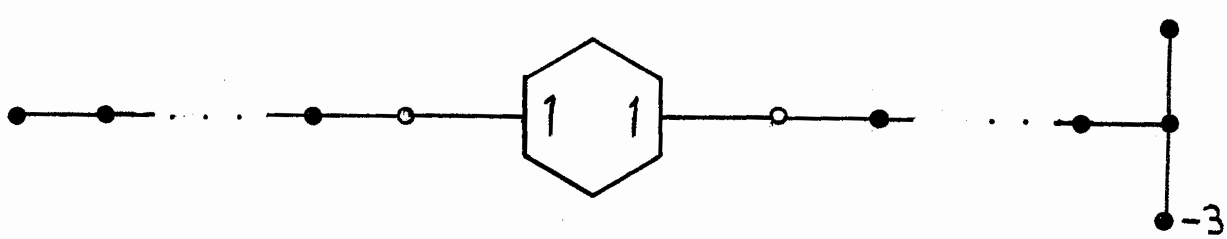
As the second component does not contain any exceptional curve, one of the arms of  is not connected to anything, so this can be suppressed from the symbol:



Another way to avoid this annoying phenomenon, is to blow up once in the second component.
Then the improvement graph looks like:



We extend our graph conventions by using the symbol ○ for a smooth rational curve with self-intersection -1.(● means a smooth rational (-2)-curve, as usual). The (-1)-curves in this example are of type 1. Such curves behave in many respects as (-2)-curves. For example, when we blow up a number of times in the special point, the graph takes the following shape:



In Chapter 3 we will study these transformations more systematically. We call them *elementary transformations.*

CHAPTER 2

THE GEOMETRIC GENUS

In this chapter we introduce an important invariant of a weakly normal surface germ (X,Σ,p): the geometric genus p_g . This invariant has many properties in common with the invariant of the same name for normal surface singularities, most notably the semicontinuity under flat deformations and its interpretation as a Hodge number of the vanishing cohomology of a smoothing. These facts express the fact that p_g should be thought of as a certain 'defect'. The semicontinuity property is very useful for classification: it makes p_g into a good measure of the complexity of the singularity.

§ 2.1 The Delta-invariant and the Geometric Genus

The genus of smooth, complete and irreducible curve C is the number $g(C) := \dim H^1(\mathcal{O}_C) = \dim H^0(\omega_C)$. Here ω_C is the sheaf of regular differentials on C, and by Serre duality both dimensions above agree (ω_C is dualizing). Classically curves were studied by a plane model $D \subset \mathbb{P}^2$, obtained for instance by a projection $C \xrightarrow{n} D$. Such a plane model usually has singularities. The question that arises here is: how can we describe the space of holomorphic 1-forms $H^0(\omega_C)$ in terms of the plane model $D \subset \mathbb{P}^2$? When we choose affine coordinates x,y such that the equation for D can be written as $f(x,y) = 0$, this question can be reformulated as: What rational differentials $\omega = R(x,y).dx$ are "everywhere finite on C" ? Such a differential is called *of the first kind.* It turns out that such a differential has to be of the form:
(2.1.1)

$$\omega = \frac{\phi(x,y).dx}{(\partial f/\partial y)}$$

where ϕ is a polynomial with $\deg(\phi) \le \deg(f) - 3$.

In addition, for a differential ω to be of the first kind it is (per definitionem) necessary and sufficient that ϕ is an *adjoint* of the curve D, or that ϕ satisfies the *adjunction conditions*, which force ϕ to vanish to sufficient high order on the singular points of D. For example, when D has only ordinary nodes and cusps as singular points, the adjoints are those curves which pass through all singular points.(For this classical theory see [C-G], [B-K].) Define the *adjunction ideal (sheaf)* by:

$$\mathcal{J} = \{ \phi \in \mathcal{O}_{\mathbb{P}^2} \mid n^*\left[\frac{\phi \, . \, dx}{(\partial f/\partial x)} \right] \in \omega_C \}$$

Here the map $C \xrightarrow{n} D$ can be considered as the normalization of D. A completely satisfactory description of this adjunction ideal was given only comparatively recent ([Go]) and is formulated most naturally in terms of *local duality*.

First, the differentials of (2.1.1) are considered as global sections of an invertible sheaf ω_D , the *dualizing sheaf* of the curve D. So ω_D is obtained by taking residues of forms on $\mathbb{P}^2$. Then $\mathcal{J}$ can be described as $\mathcal{J} = \mathrm{Ann}\,(\omega_D/n_*\omega_C)$. Now one has the following miraculous:

(2.1.2) Duality isomorphism :

$$n_*\omega_C \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_D}(n_*\mathcal{O}_C \, , \, \omega_D)$$

(see [R-R-V], [Ha 1], see also (2.2.2))

As ω_D is locally free one has $\mathcal{H}om(n_*\mathcal{O}_C \, , \, \omega_D) \approx \mathcal{H}om(n_*\mathcal{O}_D \, ,\mathcal{O}_C) \otimes \omega_D$ and $\mathcal{H}om(n^*\mathcal{O}_C \, , \, \mathcal{O}_D) = \mathrm{Ann}(n_*\mathcal{O}_C/\mathcal{O}_D)$ is just the *conductor ideal* of the map $C \xrightarrow{n} D$ (see (1.2.11)). We conclude that the adjunction ideal is the same as the conductor ideal.

(The duality isomorphism can be reformulated as the statement that for every singular point p of D the local residue pairing

$$\begin{array}{llll} R_p : (\omega_D/n_*\omega_C)_p & \times \; (n_*\mathcal{O}_C/\mathcal{O}_D)_p & \longrightarrow & \mathbb{C} \\ \quad [\omega] & , \quad [g] & \longmapsto & \Sigma_{q \in n^{-1}(p)} \mathrm{Res}_q(g.n^*\omega) \end{array}$$

is non-degenerate (see [Se]). Hence $\omega_D/n_*\omega_C$ and $n_*\mathcal{O}_C/\mathcal{O}_D$ have the same annihilator).

From this we see that the number of adjunction conditions imposed by a singular point $p \in D$ is just the δ -invariant $\delta(D,p)$ (1.2.23). Using the exact sequence

$$0 \longrightarrow n_*\omega_C \longrightarrow \omega_D \longrightarrow \mathcal{O}/\mathcal{J} \longrightarrow 0$$

and the isomorphism $\omega_D \approx \mathcal{O}_D(d-3)$, $d = \deg(D)$ one gets

(2.1.3) Genus formula :

$$g(C) = (d-1).(d-2)/2 - \delta(D)$$

The adjunction conditions on curves of degree d-3 at the different points are independent.

Of course, there are more easy ways to prove (2.1.3), but most other proofs use some kind of *deformation* argument. For instance, one can deform D to a smooth curve by perturbing the defining equation for D. This leads to the following situation:

(2.1.4) Situation :

$$\begin{tikzcd} D \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "\Pi"] \\ \{0\} \arrow[r] & S \end{tikzcd}$$

Here $\mathcal{D}$ is the surface $\{((x:y:z),t) \in \mathbb{P}^2 \times S \mid F(x,y,z,t) = 0\}$, $F(x,y,z,0)$ the projective equation for D and S a smooth curve germ, parametrized by t. Π is the evident projection map. Put $D_t = \Pi^{-1}(t)$ and assume this to be a smooth curve for $t \neq 0$. By the standard "semicontinuity theorem" (see [Ha 2]) one has:

$$\chi(\mathcal{O}_D) = \chi(\mathcal{O}_{D_t})$$

Using $0 \longrightarrow \mathcal{O}_D \longrightarrow n_*\mathcal{O}_C \longrightarrow \mathbb{C}^{\delta(D)} \longrightarrow 0$ and the definition $\chi(\mathcal{O}_C) = 1 - g(C)$ one arrives at $g(C) = g(D_t) + \delta(D)$. When one nows that the genus of a smooth plane curve of degree d is (d-1).(d-2)/2 this is equivalent with (2.1.3).
In any case the δ-invariant, originally defined as a purely local invariant of a curve singularity, appears to govern the behaviour of the genus of a curve in a *family*.

The genus of a smooth complete curve C has a clear topological meaning as $g(C) = (1/2) \dim H^1(C,\mathbb{Z})$. This suggests comparing $H^1(D,\mathbb{Z})$ with $H^1(C,\mathbb{Z})$ and $H^1(D_t,\mathbb{Z})$ in the above situation (2.1.4). One can choose a *contraction* $\rho : D_t \longrightarrow D$ of the 'nearby fibre D_t' to the singular fibre D. This leads to exact sequences: (see [Stee 1])

(2.1.5)

$$0 \longrightarrow H^1(D) \longrightarrow H^1(D_t) \longrightarrow H^1(R\Phi) \longrightarrow 0$$

(2.1.6)

$$0 \longrightarrow \oplus\mathbb{Z}^{r(p)-1} \longrightarrow H^1(D) \longrightarrow H^1(C) \longrightarrow 0$$

(where r(p) is the number of branches of D at p)

Here $H^1(R\Phi)$ is the *vanishing cohomoly group* of the family $\mathcal{D} \longrightarrow S$, that is, the first cohomology of the local Milnor fibre (see [Mi], [Lo]). Each singular point p of D contributes $\mu(D,p)$, the *Milnornumber*, to $H^1(R\Phi)$.
Now the sequences (2.1.5) and (2.1.6) are sequences of *Mixed Hodge Structures* (MHS) (see [Stee 1]): the MHS on $H^1(D)$ appears in (2.1.6) as an extension of two pure ones; the MHS on $H^1(D_t)$ is the *limes Mixed Hodge Structure* as constructed in [Stee 1]. The MHS on $H^1(R\Phi)$ is defined by the sequence (2.1.5), but is of *local nature* , i.e. depends only on the local structure of $\mathcal{D}$ around the singular points. One can compute that :

$$Gr_F^1 H^1(R\Phi) = \delta \ ; \ Gr_F^0 H^1(R\Phi) = \delta - r + 1$$

and taking Gr_F^1 of the sequences (2.1.5) and (2.1.6) one refinds formula (2.1.3). So this approach gives us a totally different interpretation of the δ-invariant as the *holomorphic part of the vanishing cohomology.*

Let us prove that δ is a *semicontinuous* invariant. By this we mean the following: consider a flat deformation $\mathcal{D} \longrightarrow S$ of a reduced curve germ (D,p) with an isolated singular point p over a smooth curve germ $(S,0)$. So the general fibre D_t, $t \neq 0$ now may have several isolated singular points, each with their own δ-invariant. By semicontinuity of δ we mean that:

$$\delta(D) \geq \delta(D_t) := \sum_{q \in D_t} \delta(D_t, q)$$

(2.1.7) Proposition : δ is a semicontinuous invariant.

proof : Let
$$\begin{tikzcd} D \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\ \{0\} \arrow[r] & S \end{tikzcd}$$
be a flat deformation of the germ D over a smooth curve germ S. So the total space is a certain surface $\mathcal{D}$, in general with a one-dimensional singular locus Σ Because we supposed D to be reduced, Σ maps finite to S. Consider the normalization $\tilde{\mathcal{D}} \xrightarrow{n} \mathcal{D}$, and put $\bar{D} = n^{-1}(D)$. One has the following diagram with exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathcal{D}} \arrow[r, "t"] \arrow[d, hook] & \mathcal{O}_{\mathcal{D}} \arrow[r] \arrow[d, hook] & \mathcal{O}_{D} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{O}_{\tilde{\mathcal{D}}} \arrow[r, "t"] \arrow[d, two heads] & \mathcal{O}_{\tilde{\mathcal{D}}} \arrow[r] \arrow[d, two heads] & \mathcal{O}_{\bar{D}} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r, "t"] & \mathcal{G} \arrow[r] & \mathcal{O}_{\bar{D}}/\mathcal{O}_{D} \arrow[r] & 0
\end{tikzcd}$$

Multiplication by t, the local parameter on S, is injective on $\mathcal{O}_{\tilde{\mathcal{D}}}$ and $\mathcal{O}_{\mathcal{D}}$ because we have a flat deformation. Injectivity of the map $\mathcal{O}_D \longrightarrow \mathcal{O}_{\bar{D}}$ follows from the fact that D was assumed to be reduced. So by the snake, t acts injectively on the sheaf $\mathcal{G}$ also. Taking the direct image, we may consider $\mathcal{G}$ as a locally free sheaf on S. On the general point of S it has rank $\delta(D_t)$ and the fibre over 0 has dimension $\dim(\mathcal{O}_{\bar{D}}/\mathcal{O}_D)$. As the curves D and $\bar{D}$ have the same normalization this dimension is also $\delta(D) - \delta(\bar{D})$. Hence we arrive at:

$$\delta(D) = \delta(\bar{D}) + \delta(D_t)$$

As $\delta(\bar{D}) \geq 0$ we get the semicontinuity of δ (and even more for example: δ constant $\Rightarrow$ $\tilde{\mathcal{D}}$ smooth). ∎

(2.1.8) Remark : This proposition is due to Teissier (see [Te]). We include a proof here, because this is somewhat like a model for the semicontinuity theorem of § 2.5.

(2.1.9) Remark : One can define the δ-invariant for a curve germ D that is generically reduced, but is allowed to have nilpotents as:

$$\delta(D) = \delta(D_{red}) - \text{Torsion}(\mathcal{O}_D)$$

(see [Stee 5]) A slight change in the proof of (2.1.7) shows that this invariant is also semicontinuous. One can use the following simple lemma:

(2.1.10) Lemma : Let $\mathcal{F} \xrightarrow{\psi} \mathcal{G}$ be a map between two locally free sheaves on a smooth curve S with local parameter t. Assume that the induced map

$$\psi_s : \mathcal{F}/(t-s).\mathcal{F} \longrightarrow \mathcal{G}/(t-s).\mathcal{G}$$

has an index for s = 0. Then ψ_s has an index (for s small) and

$$\text{Index}(\psi_0) = \text{Index}(\psi_s)$$

(2.1.11) Having discussed the case of curves at some length, we turn our attention to...surfaces.
Let Y be a smooth projective surface, and $Y \xrightarrow{\pi} X \subset \mathbb{P}^3$ a birational mapping from Y to X. The map $Y \xrightarrow{\pi} X$ can be considered as a resolution of the singularities of X. In general, the singular locus of X will be a space curve Σ, singular itself. The *geometric genus* of Y is the number $p_g := \dim H^0(\omega_Y) = \dim H^2(\mathcal{O}_Y)$ Clebsch and Noether ([Cl],[No]) started to study the *double integrals of the first kind* on X, i.e. expressions of the form: $\iint \omega$, with ω a differential of the first kind, which again turns out to have the form:

$$\omega = \frac{\phi\ .dx.dy}{(\ \partial f/\partial z)}$$

where f(x,y,z) = 0 is an affine equation of X, and ϕ is an adjoint polynomial, now with $\deg(\phi) \le \deg(f) - 4$. The adjunction conditions require ϕ to vanish to sufficient order on Σ. But in constrast to the curve case, there does not exist a direct algebraic description of the adjunction ideal

$$\mathcal{J} = \{ \phi \in \mathcal{O}_X \mid \pi^*\left[\frac{\phi.dx.dy}{(\partial f/\partial z)}\right] \in \omega_Y \} = \mathrm{Ann}_{\mathcal{O}_X}(\omega_X/\pi_*\omega_Y)$$

(so far as I know at least).

The classical approach is to split up the adjunction problem in two parts. One defines ϕ to be a *subadjoint* of X if $\phi|_H$ is an adjoint of the general hyperplane section $X \cap H$. The corresponding subadjunction ideal $\mathcal{J}_s$ is easily seen to be the annihilator of $\omega_X/n_*\omega_{\tilde{X}}$, where $\tilde{X} \xrightarrow{n} X$ is the normalisation of X. Using the duality isomorphism for finite maps (2.1.2), this is equal to $\mathrm{Ann}(n_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X)$, the conductor. This reduces the problem to the case of the isolated singularities of $\tilde{X}$ (which now however need not be hypersurface or even Gorenstein singularities). But when X is obtained from Y by a *general projection* , X has only "ordinary pinch points (D_∞)" and "ordinary triple points ($T_{\infty,\infty,\infty}$)" (see [G-H], Ch.4) as singularities. Hence, in that case the normalization $\tilde{X}$ is smooth and the adjunction ideal is the same as the subadjunction ideal. But as *isolated* singularities naturally appear in degeneration situations, we cannot be satisfied with only this knowledge.

Let us look at the role played by the isolated singular points. Each such a point $p \in X$ imposes

$$p_g(X,p) := \dim\, (\omega_X/\pi_*\omega_Y)_p$$

adjunction conditions (which, however, are not independent on polynomials of degree d - 4 in general).

This number $p_g(X,p)$ is called the *geometric genus of the isolated singular point p.*

Du Val [DuV] was the first who determined which singular points of surfaces in $\mathbb{C}^3$ do not affect the adjunction conditions, i.e. those with $p_g = 0$. These rational double points (RDP's) as they are called now, were described much earlier by Klein [Kl] as quotients of $\mathbb{C}^2$ by a finite subgroup $G \subset Sl(2,\mathbb{C})$ and the classification of these subgroups goes back to Schwartz [Schw]).

The list of RDP's coincides with the list of *simple* map germs $f:\mathbb{C}^3 \longrightarrow \mathbb{C}$ as was discovered by Arnol'd ([Arn 1], [A-G-V]) (and this is still a miracle). For several reasons these singularities carry the names of the Dynkindiagrams of type A_n $(n\geq1)$, D_n $(n\geq4)$, E_n $(n=6,7,8)$. We refer to [Dur] for an overview of the different characterizations of this remarkable family of singularities.

The geometric genus p_g can be computed with the help of the Newton diagram (see [M-T]) of f, as was discovered by Hodge (see [Hod]). In fact, Baker (see [Ba]) computed the δ-invariant for plane curves the same way already in 1894. When f is degenerate with respect to its Newton diagram there is still a description in terms of the $V^{\cdot}$-filtration of the Asymptotic Mixed Hodge Structure (see [S-S]).

The invariant p_g is, as the δ-invariant of a curve singularity, semicontinuous under flat deformations. The proof of this fact contains a new element: the *Vanishing Theorem* of Grauert and Riemenschneider (see [G-Ri]). We will discuss this in the next section.

§ 2.2 Genus and Defect

In this paragraph, we give some general facts and results that are related to the geometric genus of a singularity. (This account is mainly based on [E1].) It will be notationally convenient to work in a suitable derived category, like $D^b_c(X)$ of bounded complexes of sheaves with coherent cohomology sheaves.
By K^i we denote the i-th cohomology sheaf of a complex $K^\cdot$.

(2.2.1) Definition : Let $\pi : Y \longrightarrow X$ be a *proper* map between two (analytic) spaces X and Y.

We call the complex

$\mathcal{P}^\cdot(\pi) = \mathrm{Cone}(\mathcal{O}_X \longrightarrow R^\cdot\pi_*\mathcal{O}_Y)$

the *defect (complex)* of the map π.
A complex $K^\cdot$ is called *concentrated at p* if $\mathrm{supp}(K^i) \subset \{p\}$. A map π as above is called concentrated if its defect is. For example, all point modifications, in particular all resolutions of *isolated* singularities are concentrated maps. In two cases it is possible in a straightforward way to define the Euler characteristic of a complex: a) when X is compact or b) when the complex is concentrated. (In both cases: $\chi(K^\cdot) = \sum_i (-1)^i \mathbb{H}^i(X,K^\cdot)$, where $\mathbb{H}$ is hypercohomology.) We put $\chi(\pi) = \chi(\mathcal{P}^\cdot(\pi))$.
Note the following obvious properties:

1) $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X) + \chi(\pi)$.
2) For a composition $Z \xrightarrow{g} Y \xrightarrow{f} X$ one has:
$R^\cdot f_*\mathcal{P}^\cdot(g) = \mathrm{Cone}(\mathcal{P}^\cdot(f) \longrightarrow \mathcal{P}^\cdot(f\circ g))$.
3) In particular for the composition of concentrated maps:
$\chi(f\circ g) = \chi(f) + \chi(g)$.

Now consider a resolution $Y \xrightarrow{\pi} X$ of the singularities of X. The crucial property of the defect $\mathcal{P}^\cdot(\pi)$ is that in fact it does not depend on the resolution chosen. Let us see why this is the case.

Remember that for every space there is a *dualizing complex* $\omega^\cdot_X$ naturally attached to it (When X is of dimension n, then one has $\omega^i_X = 0$, $i \notin [-n,0]$; if X is CM, then $\omega^i_X = 0$, $i \neq -n$, $\omega_X := \omega^{-n}_X$; if X is smooth, then $\omega_X \approx \Omega^n_X$, the top differentials.)(see [R-R]).

Part of duality theory can be summarized in:

(2.2.2) Duality isomorphism: For every proper map $Y \xrightarrow{\pi} X$ there is a natural isomorphism:

$$R^{\cdot}\pi_* R^{\cdot}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}^{\cdot}, \omega^{\cdot}_Y) \xrightarrow{\sim} R^{\cdot}\mathcal{H}om_{\mathcal{O}_X}(R^{\cdot}\pi_*\mathcal{F}^{\cdot}, \omega^{\cdot}_X)$$

For a proof, see [R-R-V].

When we take $\mathcal{F}^{\cdot} = \omega^{\cdot}_Y$, we get a ("fibre integration") map:

$$R^{\cdot}\pi_*\omega^{\cdot}_Y \longrightarrow \omega^{\cdot}_X$$

We call the complex $\mathcal{Q}^{\cdot}(\pi) = \mathrm{Cone}(R^{\cdot}\pi_*\omega_Y \longrightarrow \omega^{\cdot}_X)$ the *codefect* of the map π. Using (2.2.2) it is easy to verify that:

$$\mathcal{P}^{\cdot}(\pi) = R^{\cdot}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}^{\cdot}(\pi), \omega^{\cdot}_X)$$

$$\mathcal{Q}^{\cdot}(\pi) = R^{\cdot}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^{\cdot}(\pi), \omega^{\cdot}_X)$$

Because every two resolutions are dominated by a third one and because of the composition property of the defect, it is sufficient to show $\mathcal{P}^{\cdot}(\pi) = 0$ for a bimeromorphic map between two *smooth* spaces to conclude independence of the resolution. Dually, it is sufficient to show $\mathcal{Q}^{\cdot}(\pi)$ for such a map. For this one can use:

(2.2.3) Vanishing theorem of Grauert and Riemenschneider:

Let $Y \xrightarrow{\pi} X$ be a proper bimeromorphic map between two analytic spaces. Assume that Y is *smooth*. Then one has:

$$R^i\pi_*\omega_Y = 0 \text{ for all } i > 0$$

For a proof see [G-R].

From this the independence follows: if X is smooth, then $\omega^{\cdot}_X[-n] = \omega_X = \Omega^n_X$. The inclusion $\pi_*\Omega^n_Y \longrightarrow \Omega^n_X$ is in fact an isomorphism, because regular differentials can be pulled back. So we see $\mathcal{Q}^{\cdot}(\pi) = 0$.

Now we know that $\mathcal{P}^{\cdot}(\pi)$ does not depend on the resolution π the following definitions make sense:

(2.2.4) Definitions: Let $Y \xrightarrow{\pi} X$ be a resolution of an n-dimensional space X.

1) the complex $\mathcal{P}^\cdot(X) := \mathcal{P}^\cdot(\pi)$ we call the *genus* of X. (Similarly there is a *cogenus* $\mathcal{Q}^\cdot(X) = \mathcal{Q}^\cdot(\pi)$.)

2) X is said to be *rational* at the point p if $\mathcal{P}^\cdot(X,p) := \mathcal{P}^\cdot(X)_p = 0$.

3) If X has an *isolated* singular point at p, its *geometric genus* is the number $p_g(X,p) = (-1)^{n+1}.\chi(\mathcal{P}(X,p))$.

(2.2.5) Remarks:

1) It is easy to see that X is rational at p if and only if it is *normal*, *Cohen-Macaulay* and does not effect the adjunction conditions, i.e: $\pi_*\omega_Y \approx \omega_X$ (all at p).

2) It is probably too late to change the name geometric genus into the better name arithmetic genus. The reason for the sign will be clear in a moment.

3) For an isolated purely n-dimensional singularity one has:

$$\begin{aligned} \mathcal{P}^i(X,p) &= \mathcal{H}^{i+1}_{\{p\}}(\mathcal{O}_X)\ , \quad i = -1, 0, \ldots, n-2. \\ \mathcal{P}^{n-1}(X,p) &= R^{n-1}\pi_*\mathcal{O}_Y \quad (/\mathcal{O}_X \text{ if } n = 1) \\ \mathcal{P}^j(X,p) &= 0 \quad\quad , \ j \geq n. \end{aligned}$$

4) When X is a generically reduced one-dimensional singularity, then p_g is just the δ-invariant of (1.2.23) and (2.1.9).

5) When we choose a small Stein representative for an isolated singularity of dimension $n \geq 2$ we have:

$(R^{n-1}\pi_*\mathcal{O}_Y)^* = H^0(\omega_{Y-E})/H^0(\omega_Y)$, where E is the exceptional set of the map $Y \xrightarrow{\pi} X$. So if X is CM the number $p_g(X,p)$ is the dimension of the vector space of n-forms on Y-E that cannot be extended to holomorphic ones on the whole of Y.

We now state two theorems that might illustrate the relevance of the geometric genus:

(2.2.6) **Theorem ([El])** : The invariant p_g is upper semi-continuous under flat deformations.

I.e: given a flat deformation $\mathcal{X} \xrightarrow{f} S$ of a space X with an isolated singular point p we have:

$$p_g(X,p) \geq \Sigma_{q \in X_t}\, p_g(X_t,q)$$

For a proof see [El].(There the result is only stated for *normal* X.) The main ingredient of the proof is the Grauert - Riemenschneider vanishing theorem (2.2.3) which makes it possible to do roughly the same with $\omega_{\mathcal{X}}$ as with $\mathcal{O}_{\mathcal{X}}$ in (2.1.7). We use this idea in § 2.5. Elkik in fact proves the semicontinuity of all partial Euler characteristics of $\mathcal{P}^{\cdot}(X,p)$.

For the second theorem we need a *smoothing* $\mathcal{X} \xrightarrow{f} S$ of an isolated singularity (X,p) over a smooth curve germ S. Associated to such a situation there is a Limit Mixed Hodge Structure on the cohomology groups $H^i(R\Phi) \approx H^i(X_t,\mathbb{C})$. The Hodge filtration can be described as follows: take an embedded resolution of X in $\mathcal{X}$ to obtain a space $\mathcal{Y} \xrightarrow{\pi} \mathcal{X}$. We may assume that $Y:=(f\circ\pi)^{-1}(0)$ is a normal crossing divisor, reduced after a finite base change. On the space $\mathcal{Y}$ we can consider the complex $K^{\cdot}=\Omega^{\cdot}_{\mathcal{Y}/S}(\log Y)$ of relative differential forms with a logarithmic pole along Y. Let $E = \pi^{-1}(p)$ be the compact part of Y. The complex $K^{\cdot}\otimes \mathcal{O}_E$ can be used to give $H^i(R\Phi)$ its Mixed Hodge Structure and one has:

$$Gr_F^p\, H^n(R\Phi) = H^{n-p}(E,\Omega^p_{\mathcal{Y}/S}(\log Y)\otimes\mathcal{O}_E)$$

In particular, $Gr_F^n\, H^n(R\Phi)$ can be identified with:

(2.2.7)

$$Gr_F^n\, H^n(R\Phi) = H^0(E,\omega_E(D))$$

by the adjunction formula. Here $D = E \cap \tilde{X}$, where $\tilde{X}$ is the strict transform of X in $\mathcal{Y}$ (and for non-isolated singularities the union of all non-compact components of Y).(For all these facts see [Stee 3], [Stee 5].) Now one has:

(2.2.8) Theorem ([Stee 3]) : For a smoothing $\mathcal{X} \longrightarrow S$ of an isolated singularity one has:

$$Gr_F^n H^n(R\Phi) = p_g(X,p)$$

(2.2.9) Remarks:

1) For isolated hypersurface singularities theorem (2.2.8) was proved by M. Saito [Sa].

2) For singularities with the property that $H^i(R\Phi) = 0$ for all but one value i=n (so called *spherical singularities*, for example complete intersections, see [Lo]) one has semicontinuity of all $Gr_F^p H^n(R\Phi)$, see [Stee 4].

3) One should proof (2.2.6) and (2.2.8) at the same time by proving that for a general flat deformation $\mathcal{X} \xrightarrow{f} S$ of an isolated singularity (X,p) one has:

$$p_g(X,p) - \Sigma_{q\in X_t} p_g(X_t,q) = Gr_F^n H^n(R\Phi)$$

where $H(R\Phi)$ is the vanishing cohomology of the given family.

So for isolated singularities we have a number p_g with good properties and for non-isolated singularities there is a complex in the derived category, but so far we do not have a single semicontinuous invariant for a non-isolated singularity. The problem is: How does one formulate semicontinuity properties for a *complex* ?
One such property was discovered by Elkik: general non-isolated rational singularities only deform into rational singularities.
The next step is: look at singularities for which the complex $\mathcal{P}^\cdot(X)$ is concentrated at one point p (i.e singularities which are rational outside one point). There does not seem to be any problem in proving the semicontinuity of the Euler characteristic in that case. But surfaces with non-isolated singularities are *not* of this type, so we will not pursue this any further here.
Another try might be: look at the Hodge numbers of the vanishing cohomology. But these are (of course) not always semicontinuous:

(2.2.10) Example : Let $X = \{(x,y,z) \in \mathbb{C}^3 | \ y^3 - z^2 = 0 \}$.
Then $H^2(R\Phi) = 0$ for the canonical smoothing given by the equation of X, but X deforms into isolated singularities with arbitrarily high p_g.

However, there do exist non-isolated surface singularities that deform *only* to isolated singularities with a *bounded* p_g, most notably the singularities A_∞ and D_∞. Example (2.2.10) is not weakly normal, A_∞ and D_∞ are. In the next of this chapter we shall try to convince the reader that for weakly normal surfaces a satisfactory theory of p_g does exist.

§ 2.3 Geometric Genus for AWN-surfaces

In (2.2.4) we defined the genus $\mathscr{P}^{\cdot}(X)$ of a non-isolated singularity with the help of a resolution. In the spirit of improvements (see § 1.4) one may try the following kind of 'punctual' definition of p_g :

(2.3.1) Definition : Let (X,p) be a germ of an analytic space. Let $\mathfrak{M}(X,p)$ be the directed system of all maps $Y \xrightarrow{\pi} X$ such that:

1) π is proper
2) $Y - \pi^{-1}(p) \longrightarrow X - \{p\}$ is an isomorphism (and $\pi^{-1}(p)$ is nowhere dense in Y).
3) Y is Cohen-Macaulay and smooth on a generic point of $\pi^{-1}(p)$.

So all maps π in $\mathfrak{M}(X,p)$ are concentrated by construction. We put

$$P(\pi) := (-1)^{n+1}.\chi(\pi) \qquad (n = \dim(X,p))$$

$$p_g(X,p) := \lim \ \{ P(\pi) \mid \pi \in \mathfrak{M}(X,p) \}$$

(2.3.2) Remark : It is totally unclear in what sense this limit exists. For example, $\mathfrak{M}(X,p)$ could be empty. Something like condition 3) is really needed to give this limit a chance. In any case, it is clear that for an isolated singular point this definition agrees with (2.2.4), the limit being attained by a resolution.

For surface germs we can compute this limit:

(2.3.3) Theorem : Let (X,p) a purely two-dimensional germ with $X - \{p\}$ reduced. Then the limit of (2.3.1) exists and is INFINITE when $X - \{p\}$ is NOT weakly normal.

proof : Let $Y \xrightarrow{\pi} X$ be a modification in $\mathfrak{M}(X,p)$ and let the $Z \xrightarrow{q} Y$ be such that $\pi \circ q \in \mathfrak{M}(X,p)$. By assumption 3) the map $Z \xrightarrow{q} Y$ is a modification in a finite number of points, lying in $\pi^{-1}(p)$. As Y is assumed to by CM, we have $q_*\mathcal{O}_Z = \mathcal{O}_Y$. A little computation now learns that:

$$P(\pi\circ q) = P(\pi) + \dim H^0(Y, R^1q_*\mathcal{O}_Z)$$

Hence $P(\pi)$ is monotonically non-decreasing for the partial ordering given by the domination relation.
Now given a surface germ (X,p), we can always find an improvement in the sense of (1.4.2) (see remark (1.4.5)). So we assume that Y has a smooth one-dimensional singular locus Σ, that the normalization $\tilde{Y}$ is smooth, and that $\tilde{\Sigma}$ is smooth. We now give Σ and $\tilde{\Sigma}$ the possibly non-reduced structure of the conductor (1.2.11) of the map $\tilde{Y} \longrightarrow Y$, so that Y is a push-out (1.2.2). As Y has depth 2, Σ and $\tilde{\Sigma}$ are CM (no embedded components) by (1.2.12).
When $X - \{p\}$ is **not** WN, it follows from (1.2.20) that there is a component of $\tilde{Y}$ on which $\tilde{\Sigma}$ is not a reduced curve. Choose local coordinates x,y in this component, such that $x^c = 0$, $c \geq 2$ is an equation for $\tilde{\Sigma}$.
Now blow up in the point $(0,0)$. The resulting space we call $\tilde{Z}$. Let $\tilde{\Delta}$ be the (non-reduced) strict transform of $\tilde{\Sigma}$ in $\tilde{Z}$. The map $\tilde{\Delta} \longrightarrow \tilde{\Sigma}$ induces an inclusion $\mathcal{O}_{\tilde{\Sigma}} \longrightarrow \mathcal{O}_{\tilde{\Delta}}$ which in coordinates looks like:

$$\mathbb{C}\{x,y\}/(x^c) \longrightarrow \mathbb{C}\{u,v\}/(u^c) \;;\; x = u.v \;,\; y = v$$

Now form the push-out Z on $\tilde{\Delta} \longrightarrow \tilde{Z}$ and $\tilde{\Delta} \longrightarrow \Sigma$ (via $\tilde{\Delta} \longrightarrow \tilde{\Sigma} \longrightarrow \Sigma$) as in (1.2.6) . We get an induced map $q\colon Z \longrightarrow Y$ by the universal property. In general Z will not be Cohen-Macaulay, but because Z is a surface, we can give it depth 2 (see(1.1.4)). Let $\hat{Z} \xrightarrow{c} Z$ be the CM-ification map. Note the exact sequences of glueing:

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow n_*\mathcal{O}_{\tilde{Y}} \longrightarrow n_*\mathcal{O}_{\tilde{\Sigma}}/\mathcal{O}_\Sigma \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow n_*\mathcal{O}_{\tilde{Z}} \longrightarrow n_*\mathcal{O}_{\tilde{\Delta}}/\mathcal{O}_\Sigma \longrightarrow 0$$

Taking q_* of the second sequence, and using $R^1q_*n_*\mathcal{O}_{\tilde{Z}} = 0$, $q_*\mathcal{O}_Z=\mathcal{O}_Y$ $q_*n_*\mathcal{O}_{\tilde{Z}} = n_*\mathcal{O}_{\tilde{Y}}$ and combining it with the first sequence gives:

$$0 \longrightarrow n_*\mathcal{O}_{\tilde{\Sigma}}/\mathcal{O}_\Sigma \longrightarrow n_*\mathcal{O}_{\tilde{\Delta}}/\mathcal{O}_\Sigma \longrightarrow R^1q_*\mathcal{O}_Z \longrightarrow 0$$

So $\dim R^1q_*\mathcal{O}_Z > 0$, but this does not quite proof what we want, because Z need not be CM. We claim however that also $R^1q_*c_*\mathcal{O}_{\hat{Z}} \neq 0$. This can be seen as follows: Take q_* of the sequence

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow c_*\mathcal{O}_{\hat{Z}} \longrightarrow \mathcal{C} \longrightarrow 0\ ,\ \dim(\mathcal{C}) < \infty$$

gives $0 \longrightarrow \mathcal{C} \longrightarrow R^1q_*\mathcal{O}_Z \longrightarrow R^1q_*c_*\mathcal{O}_{\hat{Z}} \longrightarrow 0$

So we have to exhibit an element in $R^1q_*\mathcal{O}_Z \approx \mathcal{O}_{\tilde{\Delta}}/\mathcal{O}_{\tilde{\Sigma}}$ which is not in $\mathcal{C}$. As in the proof of (1.2.22) one sees that the elements of $\mathcal{C}$ correspond to torsion elements in $n_*\mathcal{O}_{\tilde{\Delta}}/\mathcal{O}_{\Sigma}$.

Consider the element $u \in \mathcal{O}_{\tilde{\Delta}}$, in the coordinates as above. It is clear that $u \notin \mathcal{O}_{\tilde{\Sigma}}$, so u represents a non-zero element in $R^1q_*\mathcal{O}_Z$. Further, the class of u in $n_*\mathcal{O}_{\tilde{\Delta}}/\mathcal{O}_{\Sigma}$ cannot be torsion. Namely, assume $v^m.u \in \mathcal{O}_{\Sigma}$ for $m \gg 0$, but $v^m.u = y^{m-1}.x$, meaning that the class of x is torsion in $n_*\mathcal{O}_{\tilde{\Sigma}}/\mathcal{O}_{\Sigma}$. As Y was CM, it would follow that $x \in \mathcal{O}_{\Sigma}$, and consequently, the conductor would be reduced, contrary to our original assumption.

So starting from a modification $Y \xrightarrow{\pi} X$ we constructed another one, namely $\hat{Z} \xrightarrow{\pi\circ q\circ c} X$ with $P(\pi\circ q\circ c) > P(\pi)$. Iterating this construction shows that the limit is infinite, as soon as $X - \{p\}$ is not weakly normal. ∎

(2.3.4) Remark : Without the condition 3) on $\mathfrak{M}(X,p)$ there may very well exist chains of modifications along which $P(\pi)$ is decreasing.

Theorem (2.3.3) already shows the special role played by the AWN-surface germs. For these surfaces we have:

(2.3.5) Theorem : Let (X,Σ,p) be an AWN-surface germ. Then $p_g(X,p) = \lim \{ P(\pi) \mid \pi \in \mathfrak{M}(X,p) \}$ is FINITE.

The limit is attained for every improvement $Y \xrightarrow{\pi} X$.

Further, if $\tilde{X} \xrightarrow{n} X$ is the normalization map, $\Sigma = \mathrm{Sing}(X)$, $\tilde{\Sigma} = n^{-1}(\Sigma)$, taken with their conductor structure, then:

$$p_g(X,p) = p_g(\tilde{X}) + \delta(\tilde{\Sigma}) - \delta(\Sigma)$$

where the δ-invariant is as in (2.1.9).

proof : As improvements are cofinal in $\mathfrak{M}(X,p)$, we only have to prove $P(\pi) = p_g(\tilde{X}) + \delta(\tilde{\Sigma}) - \delta(\Sigma)$ for one improvement $Y \xrightarrow{\pi} X$.

We use the notation as in § 1.2 and (1.4.7).
There is a pair of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}_X \arrow[r] \arrow[d, "\wr"] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_{\tilde{X}} \arrow[r] \arrow[d] & \mathcal{S}_\Sigma \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathcal{T}_X \arrow[r] & \mathcal{O}_\Sigma \arrow[r] & \mathcal{O}_{\tilde{\Sigma}} \arrow[r] & \mathcal{S}_\Sigma \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{T}_X = \mathcal{H}^0_{\{p\}}(\mathcal{O}_X)$. Similarly for Y:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_Y \arrow[r] \arrow[d] & \mathcal{O}_{\tilde{Y}} \arrow[r] \arrow[d] & \mathcal{S}_\Delta \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathcal{O}_\Delta \arrow[r] & \mathcal{O}_{\tilde{\Delta}} \arrow[r] & \mathcal{S}_\Delta \arrow[r] & 0
\end{tikzcd}$$

There is mapping from the first diagram to $R\pi_*$ of the second diagram, giving rise to :

$$\begin{tikzcd}
0 \arrow[r] & \pi_*\mathcal{O}_Y \arrow[r] & \pi_*\mathcal{O}_{\tilde{Y}} \arrow[r] & \pi_*\mathcal{S}_\Delta \arrow[r] & R^1\pi_*\mathcal{O}_Y \arrow[r] & R^1\pi_*\mathcal{O}_{\tilde{Y}} \arrow[r] & 0 \\
\mathcal{T}_X \arrow[r, hook] \arrow[u] & \mathcal{O}_X \arrow[r] \arrow[u] & \mathcal{O}_{\tilde{\Sigma}} \arrow[r] \arrow[u] & \mathcal{S}_\Sigma \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \pi_*\mathcal{O}_\Delta \arrow[r] & \pi_*\mathcal{O}_{\tilde{\Delta}} \arrow[r] & \pi_*\mathcal{S}_\Delta \arrow[r] & 0 \\
\mathcal{T}_X \arrow[r, hook] & \mathcal{O}_\Sigma \arrow[r] \arrow[u] & \mathcal{O}_{\tilde{\Sigma}} \arrow[r] \arrow[u] & \mathcal{S}_\Sigma \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

both with exact rows (we suppressed some maps, like n_*, from the notation).
Taking the alternating sum of the indices of the vertical maps gives:

$$\dim R^1\pi_*\mathcal{O}_Y = \dim R^1\pi_*\mathcal{O}_{\tilde{Y}} + \dim \mathcal{H}^1_{\{p\}}(\mathcal{O}_X) + \operatorname{Index}(\mathcal{S}_\Sigma \longrightarrow \pi_*\mathcal{S}_\Delta)$$

and

$$\operatorname{Index}(\mathcal{S}_\Sigma \longrightarrow \pi_*\mathcal{S}_\Delta) = \delta(\tilde{\Sigma}) - \delta(\Sigma) - \dim \mathcal{H}^0_{\{p\}}(\mathcal{O}_X)$$

Combining this we find:

$$P(\pi) := \dim R^1\pi_*\mathcal{O}_Y + \dim \mathcal{H}^0_{\{p\}}(\mathcal{O}_X) - \dim \mathcal{H}^1_{\{p\}}$$

$$= \dim R^1\pi_*\mathcal{O}_{\tilde{Y}} + \delta(\tilde{\Sigma}) - \delta(\Sigma)$$

$$= p_g(\tilde{X}) + \delta(\tilde{\Sigma}) - \delta(\Sigma)$$

The theorem is proved. ∎

(2.3.6) Remark : If X happens to be weakly normal at p, the curves Σ and $\tilde{\Sigma}$ are reduced, even at p (see (1.2.13)). If X is WNCM, then $\delta(\tilde{\Sigma}) \geq \delta(\Sigma)$ (see (1.2.24)), so in that case $p_g(X,p) \geq p_g(\tilde{X})$.
This suggests the following approach to the problem of classifying WNCM-surfaces with low values of p_g:

1) Classify finite mappings of curves $\tilde{\Sigma} \longrightarrow \Sigma$ with

   a. $\mathcal{H}^0_{\{p\}}(\mathcal{O}_{\tilde{\Sigma}}/\mathcal{O}_{\Sigma}) = 0$

   b. $\delta(\tilde{\Sigma})-\delta(\Sigma)$ small

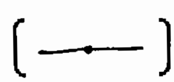
2) Classify embeddings $\tilde{\Sigma} \longrightarrow \tilde{X}$ with $p_g(\tilde{X})$ small.

In chapter 4 we make a start with this program.

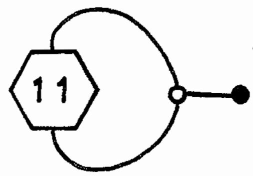
(2.3.7) Example : The partition singularities of § 1.3 have $p_g(X,p) = 0$.
In (1.2.25) we constructed certain WNCM-surfaces with A_1 as transversal singularity, obtained from $\mathbb{C}^2$ by identification with $\mathbb{Z}/2$ - action on a curve. For these it is very easy to compute the invariant p_g. We pick out one particular type of these to illustrate some interesting points:

(2.3.8) Example : We take a special look at the following two examples:

A. $\tilde{\Sigma}$ =  $x,y \longrightarrow -x,-y$, Σ =  $p_g = 2 - 1 = 1$

B. $\tilde{\Sigma}$ =  $x,y \longrightarrow x,-y$, Σ =  $p_g = 2 - 0 = 2$

The first singularity is non-Gorenstein, with embedding dimension 5. The second one is a hypersurface.
In both cases the minimal good improvement (1.4.12) looks like:



The difference between A and B is due to a difference in the identification map between the discs, making up $\tilde{A} \subset \tilde{Y}$. One can imagine this identification to change continuously from A to B. The question arises: what happens to the singularity when we do this ?
We take up a bit more general example:
$\tilde{X} = \mathbb{C}^2$, $\tilde{\Sigma} = \{(x,y) \in \mathbb{C}^2 \mid y^2 = x^{2k}\}$. We want to glue the two branches of $\tilde{\Sigma}$ together. This can be done as follows: Consider

$$\psi_1 : \mathbb{C} \longrightarrow \mathbb{C}^2 \; ; \; s \longmapsto (s, s^k)$$

$$\psi_2 : \mathbb{C} \longrightarrow \mathbb{C}^2 \; ; \; s \longmapsto (\lambda.s, -\lambda^k.s^k) \qquad \lambda \in \mathbb{C}^*$$

So the ψ_i map to the two components of $\tilde{\Sigma}$.
The ring $R = \mathcal{O}_X$ of functions on the space X we are looking for consists of those polynomials $P = P(x,y)$ with the property that $\psi_1^* P = \psi_2^* P$. Clearly, every function in the ideal $(y^2 - x^{2k})$ of $\tilde{\Sigma}$ fulfils this condition. Modulo this ideal every function has a unique representation as $P = A(x) + B(x).y$.
When we write $A(x) = \Sigma_{n\geq 1} a_n.x^n$, $B(x) = \Sigma_{n\geq 0} b_n.x^n$. The condition $\psi_1^* P = \psi_2^* P$ is now equivalent to the following system of equations for the the coefficients a_n and b_n :

1. $(\lambda^i - 1).a_i = 0 \qquad i = 1,2,\ldots,k-1$

2. $(\lambda^{k+i}-1).a_{i+k} = (\lambda^{k+i}+1).b_i \qquad i = 0,1,2,\ldots..$

When we introduce auxiliary polynomials ψ by the formula:

$$\psi(m+k) = (\lambda^{m+k} + 1).x^{m+k} + (\lambda^{m+k} - 1).x^{m}.y \qquad m = 0,1,\ldots.$$

one sees that the ψ's represent a basis for the solutions to the second set of the equations above. It is easy to verify that one has the following identity:

$I(m,l)$:

$$\psi(m+k).\psi(l+k) = 2.\psi(m+l+2k) + x^{m+l}.F.(\lambda^{k+m} - 1).(\lambda^{k+l} - 1)$$

where $F = y^2 - x^{2k}$.
From this it follows that the ψ's generate inside the ring $\mathcal{O}_{\tilde{\Sigma}} = \mathbb{C}[x,y]/(y^2 - x^{2k})$ an algebra isomorphic to

$$\mathbb{C}[s^k, s^{k+1}, s^{k+2}, \ldots] \subset \mathbb{C}[s]$$

If λ is not a p-th root of unity for $p = 1,2,\ldots,k-1$, then no a_i can satisfy the first set of equations, so then the ψ's are all. In that case $\mathcal{O}_\Sigma = \mathbb{C}[s^k, s^{k+1}, s^{k+2}, \ldots]$, so:

$$p_g(X) = \delta(\tilde{\Sigma}) - \delta(\Sigma) = k - (k-1) = 1$$

(One can check that $\mathcal{O}_{\tilde{\Sigma}}/\mathcal{O}_\Sigma$ is torsion free, so X is CM.)
To find generators for the ring R, we can apply (1.2.9) which garanties that R (strictly speaking: a completion of R) is generated by:

$\psi(k), \psi(k+1), \ldots, \psi(2k-1)$

$F,\ x.F, \ldots,\ x^{k-1}.F,\ (y+x^k).F$

However, we do not need all these generators:

lemma : If $\lambda \neq 1$, $k \geq 4$, then $x^2.F, \ldots, x^k.F \in \mathbb{C}[\psi(k), \ldots, \psi(2k-1)]$

proof : For $\lambda = -1$, this is trivial. The above identities $I(m,m+2)$ and $I(m,m+3)$ show that $x^{2(m+1)}.F$ and $x^{2(m+1)+1}.F$ are in the algebra generated by the ψ's if $\lambda^2 \neq 1$ ∎

Corollary : If $\lambda^p \neq 1$, $p=1,2,\ldots,k-1$ $(k\geq 4)$, the ring of X is generated by F, $x.F$, $y.F$, $\psi(k)$, $\psi(k+1), \ldots, \psi(2k-1)$, so we get an embedding $X \longrightarrow \mathbb{C}^{k+3}$ (minimal by reasons of degree).

If λ specializes to a root of unity, the situation changes drastically. The most degenerate case is $\lambda = 1$. In that case we find $\mathcal{O}_\Sigma = \mathbb{C}[x]$, $\mathcal{O}_\Sigma$-basis for $\mathcal{O}_{\tilde\Sigma}$: 1, y, so the ring is generated by x, F and y.F or, x, y^2 and y.F, so X is a hypersurface. For the geometric genus one finds: $p_g = \delta(\tilde\Sigma) - \delta(\Sigma) = k - 0 = k$.

Consider now a small disc Λ in the λ - plane around the point $\lambda = 1$, not containing p-th roots of unity for $p = 1,2,\ldots,k-1$. For every $\lambda \in \mathbb{C}^*$ we constructed above a space X_λ. For $\lambda = 1$ it is a hypersurface, for $\lambda \in \Lambda^* := \Lambda\setminus\{1\}$ it is not. So the family $\{X_\lambda | \lambda \in \Lambda\} \longrightarrow \Lambda$ does not represent a flat deformation of X_1. But consider the following family:

$$\mathcal{X} := \overline{\{(x,\lambda) \in \mathbb{C}^{k+3} \times \Lambda^* \mid x \in X_\lambda\}} \longrightarrow \Lambda$$

Let Y be the fibre over $\lambda = 1$ of this family. Its ring is generated by:

F, x.F, y.F, x^k, $x^{k+1}, \ldots, x^{2k-1}$

Clearly this ring is not CM. The CM-ification is just the ring of the space X_1; the 'missing functions' are $x, x^2, \ldots, x^{k-1}$. So we get $p_g(Y) = p_g(X_1) - \delta(Y) = k - (k-1) = 1$. So the family $\mathcal{X} \longrightarrow \Lambda$ has constant p_g (as it should be in a family with simultaneous improvement).

There are several other remarkable cases when λ specializes to a root of unity.

lemma : Assume that λ is a primitive p-th root of unity. Then:
$x^p.\psi(k+m) = \psi(k+m+p)$

If $p \nmid k$ then F, $y.F \in \mathbb{C}[\psi(k),\ldots,\psi(2k-1)]$

If $p \nmid k,k+1$ then $x.F \in \mathbb{C}[\psi(k),\ldots,\psi(2k-1)]$

proof : The first statement is trivial. By I(0,0) we have $\psi(k)^2 - (\lambda^k - 1)^2.F = 2.\psi(2k) = 2.x^p.\psi(2k-p)$, hence F is in the algebra generated by the ψ's. The rest is similar ∎

Corollary : If λ is a primitive p-th root of unity and $p \nmid k, k+1$ then the ring of X is generated by:

x^p, $\psi(m)$, $m \in [k, 2k-1]$ & $p \nmid m$

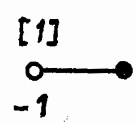
So in that case the curve Σ spans the space in which X can be embedded minimally.

The most remarkable case is when $\lambda^3 = 1$. If $3 \nmid k, k+1$ then $k = 3m-2$ for some m, so we get a hypersurface ring, generated by:

$$x^3 , \quad (\lambda + 1).x^{3m-2} + (\lambda - 1).y , \quad (\lambda^2 + 1)x^{3m-1} + (\lambda^2 - 1).x.y$$

This is Mond's simple singularity of type H_m (see (1.2.26), [Mo]) in slightly different coordinates. For this singularity one has $p_g(H_k) = \delta(\tilde{\Sigma}) - \delta(\Sigma) = (3k-2) - 2(k-1) = k$.

J. Stevens remarked that these singularities should be compared with the isolated singularities with resolution graph:



The parameter λ corresponds to the normal bundle of the elliptic curve in the resolving surface.

The Dualizing Sheaf

Let Y be a weakly smooth surface (1.4.4), so Y is assumed to have only partition singularities (1.3.1). We want to describe the dualizing sheaf of Y.
We start with a more general situation:

(2.4.1) **Proposition :** Consider a push-out diagram as in § 1.2 :

$$\begin{tikzcd} \tilde{\Sigma} \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "n"] \\ \Sigma \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & X \end{tikzcd}$$

Assume that all four spaces are Cohen-Macaulay and that $\tilde{\Sigma}$ is of pure codimension 1 in $\tilde{X}$. Then there is an exact sequence of sheaves on X:

$$0 \longrightarrow n_*\omega_{\tilde{X}} \longrightarrow \omega_X \longrightarrow n_*\omega_{\tilde{\Sigma}} \longrightarrow \omega_\Sigma \longrightarrow 0$$

where ω denotes the dualizing sheaf.

proof : Consider the exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_{\tilde{\Sigma}} \longrightarrow 0$$

and apply $\mathcal{H}om(- , \omega_{\tilde{X}})$ to this to get:

$$0 \longrightarrow \omega_{\tilde{X}} \longrightarrow \mathcal{H}om_{\mathcal{O}_{\tilde{X}}}(\mathcal{J}, \omega_{\tilde{X}}) \longrightarrow \omega_{\tilde{\Sigma}} \longrightarrow 0$$

because $\omega_{\tilde{\Sigma}} = \mathcal{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{\tilde{\Sigma}}, \omega_{\tilde{X}})$.
Similarly, we get from $\mathcal{J} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_\Sigma$ the sequence:

$$0 \longrightarrow \omega_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}, \omega_X) \longrightarrow \omega_\Sigma \longrightarrow 0$$

The map $n\colon \tilde{X} \longrightarrow X$ induces a map between these sequences:

(2.4.2) **Diagram :**

$$\begin{array}{ccccccccc} 0 & \longrightarrow & n_*\omega_{\tilde{X}} & \longrightarrow & n_*\mathcal{H}om_{\mathcal{O}_{\tilde{X}}}(\mathcal{I},\omega_{\tilde{X}}) & \longrightarrow & n_*\omega_{\tilde{\Sigma}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\wr & & \downarrow & & \\ 0 & \longrightarrow & \omega_X & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I},\omega_X) & \longrightarrow & \omega_\Sigma & \longrightarrow & 0 \end{array}$$

The middle map is an isomorphism by the duality morphism (2.2.2) (note that we deliberately confused $\mathcal{I}$ with $n_*\mathcal{I}$). By the snake lemma the proposition now follows. ∎

(2.4.3) **Remark :** When we apply $\mathcal{H}om_{\mathcal{O}_\Sigma}(-,\omega_\Sigma)$ to the sequence

$$0 \longrightarrow \mathcal{O}_\Sigma \longrightarrow n_*\mathcal{O}_{\tilde{\Sigma}} \longrightarrow \mathcal{G}_\Sigma \longrightarrow 0$$

we end up with a short exact sequence:

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_\Sigma}(\mathcal{G}_\Sigma,\omega_\Sigma) \longrightarrow n_*\omega_{\tilde{\Sigma}} \longrightarrow \omega_\Sigma \longrightarrow 0$$

(2.4.4) **Remark :** If $\tilde{\Sigma}$ is a Cartier divisor on $\tilde{X}$ then one has $\mathcal{I} = \mathcal{O}_{\tilde{X}}(-\tilde{\Sigma})$, so in that case:

$$\mathcal{H}om_{\mathcal{O}_{\tilde{X}}}(\mathcal{I},\omega_{\tilde{X}}) = \omega_{\tilde{X}}(\tilde{\Sigma})$$

It is tempting to **define** $\omega_{\tilde{X}}(\tilde{\Sigma})$ by this equation, even if $\tilde{\Sigma}$ is not Cartier.

The exact sequence of (2.4.1) thus can be derived without knowing what the dualizing sheaf is ! We however always will think about ω_X as a sheaf of certain differential forms. As an illustration, we consider the case of a partition singularity. For simplicity we take the case $X = X_{(n)}$ and use the coordinates as in (1.3.4) : $x = u^n$; $y_i = u^i.v$, $i = 0, 1, \ldots, n-1$.
The sequence:

$$0 \longrightarrow \omega_{\tilde{X}} \longrightarrow \omega_{\tilde{X}}(\tilde{\Sigma}) \xrightarrow{\text{Res}} \omega_{\tilde{\Sigma}} \longrightarrow 0$$

$$P(u,v)du\wedge\frac{dv}{v} \longmapsto P(u,0).du$$

expresses that elements of $\omega_{\tilde{\Sigma}}$ are *residues* of forms on $\tilde{X}$ with a simple pole along $\tilde{\Sigma}$.

The map $n_*\omega_{\tilde\Sigma} \longrightarrow \omega_\Sigma$ is the *trace map*:

$$u^k.du \longmapsto 0 \qquad k = 0, 1, \ldots, n-2$$

$$u^{n-1}.du \longmapsto \frac{dx}{n}$$

Generators for ω_X as an $\mathcal{O}_X$ - module thus can be represented as :

$$\frac{du\wedge dv}{v}\ ,\quad \frac{udu\wedge dv}{v}\ ,\quad \frac{u^2du\wedge dv}{v}\ ,\quad \ldots\ldots,\quad \frac{u^{n-2}du\wedge dv}{v}$$

The sequence $0 \longrightarrow \omega_X \longrightarrow n_*\omega_{\tilde X}(\tilde\Sigma) \longrightarrow \omega_\Sigma \longrightarrow 0$

expresses the fact that ω_X can be thought of as differential forms with a simple pole along Σ with the property that the sum of the residues over a fibre of $\tilde\Sigma \longrightarrow \Sigma$ is zero.
The exactness of the sequence (2.4.1) is now obvious.

(2.4.5) Proposition : " Grauert-Riemenschneider for Improvements " : Let $Y \xrightarrow{\pi} X$ be an improvement in the sense of (1.4.2).(Δ reduced)

Then : $\qquad R^i\pi_*\omega_Y = 0 \quad$ for $i \geq 1$

proof : According to (2.4.1) and (2.4.3) there are exact sequences:

$$0 \longrightarrow n_*\omega_{\tilde Y} \longrightarrow \omega_Y \longrightarrow \mathcal{H}om\,(\mathcal{G}_\Delta, \omega_\Delta) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{H}om\,(\mathcal{G}_\Delta, \omega_\Delta) \longrightarrow n_*\omega_{\tilde\Delta} \longrightarrow \omega_\Delta \longrightarrow 0$$

(we use the standard notation (1.4.7))
By ordinary Grauert-Riemenschneider for $\tilde Y$, $\tilde\Delta$ and Δ we have :

$$R^i\pi_*n_*\omega_{\tilde Y} = 0\ ,\ R^i\pi_*n_*\omega_{\tilde\Delta} = 0\ ,\ R^i\pi_*\omega_\Delta = 0 \quad \text{for} \quad i \geq 1$$

We are done by $R\pi_*$ of the above sequences, when we know that $R^1\pi_*\mathcal{H}om\,(\mathcal{G}_\Delta, \omega_\Delta) = 0$. But because Δ was assumed to be smooth, the sequence of (2.4.3) is *split* (c.f. (1.2.18)) so the result follows. ■

(2.4.6) Corollary : If (X,p) is WNCM-surface germ, then:

$$p_g(X,p) = \dim\,(\omega_X/\pi_*\omega_Y)_p$$

where $Y \xrightarrow{\pi} X$ is any improvement.

proof : This is a formal consequence of $R^1\pi_*\omega_Y = 0$ ■

§ 2.5 Semicontinuity of the Geometric Genus

For every almost weakly normal surface (X,Σ,p) we have defined a number p_g and called it the geometric genus. So far it only is a number and might be of limited relevance. The purpose of this paragraph is to show that it has at least one interesting property: p_g is semicontinuous under every flat deformation over a smooth curve germ.

(2.5.1) **Definition :** An AWN-surface germ (X,p) is called **weakly rational** if and only if $p_g(X,p) = 0$, where p_g is the invariant defined in (2.3.1)

(2.5.2) **Remark :** Even for isolated singularities this condition is really weaker than rationality in the sense of (2.2.4), so also in that case it seems to be a proper name.

Let (X,p) be a germ of an analytic space. We consider flat deformations of X over a smooth curve germ $(S,0)$.
This means that there is a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \{0\} \arrow[r] & S \end{tikzcd}$$

with f a flat map and $X = f^{-1}(0)$.
We simply say that $\mathcal{X} \xrightarrow{f} S$ is a flat deformation of X and we call $f^{-1}(t)$, $t \neq 0$ the *general fibre* of the family.

(2.5.3) **Lemma :** Let $\mathcal{X} \xrightarrow{f} S$ be a flat deformation of a space X. If X is WN outside a set of codimension $\geq d$, then the same is true for $\mathcal{X}$.

proof : Consider the weak normalization map $w : \bar{\mathcal{X}} \longrightarrow \mathcal{X}$.
The locus of non-weakly normal points of $\mathcal{X}$ is the analytic set $M :=$ support $(w_*\mathcal{O}_{\bar{\mathcal{X}}}/\mathcal{O}_{\mathcal{X}})$. There is a 'multiplication by t' diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathcal{X}} & \xrightarrow{t} & \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & w_*\mathcal{O}_{\bar{\mathcal{X}}} & \xrightarrow{t} & w_*\mathcal{O}_{\bar{\mathcal{X}}} & \longrightarrow & w_*\mathcal{O}_{X'} & \longrightarrow & 0
\end{array}$$

Here t denotes multiplication by t, a local parameter on S and $X' = w^{-1}(X)$. As $\bar{\mathcal{X}} \longrightarrow \mathcal{X}$ is a homeomorphism, $X' \longrightarrow X$ is also a homeomorphism. Hence the set of points where $X' \longrightarrow X$ is not an isomorphism, is contained in the set N of non-WN points of X. Hence $M \cap X \subseteq N$, from which the lemma follows. ∎

(2.5.4) **Corollary :** If X is WN, then $\mathcal{X}$ also is. If $X - \{p\}$ is WN, then $\mathcal{X}$ is WN outside a curve, and the nearby fibres X_t are WN outside a finite set.

We thus see that our class of AWN-surface germs (X,p) is closed under (small) flat deformations.

(2.5.5) Now we take a closer look at the general structure of the total space $\mathcal{X}$ of a flat deformation of such a surface (X,p). One can distinguish (at least) three types of deformations:

1) The fibres X_t ,$t \neq 0$, are all smooth. In this situation it is usual to call $\mathcal{X} \xrightarrow{f} S$ a *smoothing* of X. The singular locus Ξ of $\mathcal{X}$ is contained in the singular locus of Σ of X.

2) The fibres X_t ,$t \neq 0$, all have isolated singularities. The singular locus Ξ of $\mathcal{X}$ is one-dimensional and contains two kinds of components: the horizontal components Ξ_h, lying finitely over the curve S and consisting of the singularities of the fibres X_t; the vertical components Ξ_v lying in X and consisting of components of Σ.

3) The fibres X_t ,$t \neq 0$, all have non-isolated singularities, besides possibly some isolated ones. The singular locus Ξ of $\mathcal{X}$ is now two-dimensional. It consists of a purely two-dimensional part T, sweeping out the non-isolated singularities in the fibres and cutting X in a number of components of Σ. We put $\Sigma_T = T \cap X$.

Further we have as under 2) curves Ξ_h and Ξ_v, so we can write: $\Xi = T \cup \Xi_v \cup \Xi_h$. Note that $T \cap \Xi_v = T \cap \Xi_h = \{p\}$ (in general). The curves Ξ_v and Ξ_h are in general singular at the point p. In general T is singular along Σ_T and further along a curve lying finite over S. This curve is contained in the union of the special isolated points of the fibres.

From this point of view there is a clear distinction between two types of components of Σ: $\Sigma = \Sigma_T \cup \Sigma_R$ with:

$\Sigma_T = \Sigma \cap X$. These are the components of Σ that sTay during the deformation. $\mathcal{X}$ is not normal at a general point of Σ_T.

Σ_R , the other components, which are Removed under the deformation. $\mathcal{X}$ is normal at a general point of Σ_R.

In general Σ_R makes up the vertical part Ξ_v of the singular locus of $\mathcal{X}$.(Only in the case of transversal A_1 singularities $\mathcal{X}$ might be smooth along Σ.)

We shall see in a moment that from the point of view of improvements the distinction between Σ_T and Σ_R is not very important.

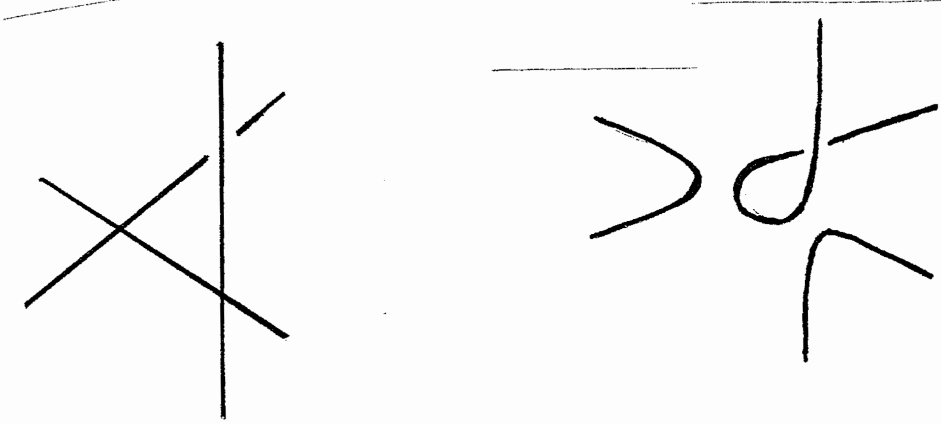
(2.5.6) We begin with a study of the singularities of $\mathcal{X}$ transverse to Σ. So let $q \in \Sigma - \{p\}$ be a general point of Σ and choose a hyperplane H through q, transverse to Σ. Thus we get a surface $\mathcal{V} := H \cap \mathcal{X}$ together with a map $\mathcal{V} \longrightarrow S$, which we again denote by f. As flatness is a local property, we may consider $\mathcal{V} \xrightarrow{f} S$ as a flat deformation of $V_0 = X \cap H$ (with reduced structure). Remember that because $X - \{p\}$ is WN , the curve V_0 is isomorphic to L_r^r for some r (see (1.1.5)).

(2.5.7) Proposition : The space $\mathcal{V}$ is WNCM and weakly rational.

proof : The weak normality follows from (2.5.3). $\mathcal{V}$ is Cohen-Macaulay because it is the total space of a flat deformation of the CM curve V_0. The weak rationality follows from a more general result that will be proved in chapter 4 (see (4.4.6)). ∎

(2.5.8) Corollary : The fibres V_t are all WN curves, so only can have L_s^s singularities. The normalization $\tilde{\mathcal{V}}$ consists of a disjoint union of isolated rational singularities, which have an L_s^s - singularity as general hyperplane section. From this it follows that on the minimal resolution of $\tilde{\mathcal{V}}$ the fundamental cycle (see § 3.4) is reduced.

(2.5.9) Example : Consider the following general fibres of two different deformations of the curve L_3^3:



The first picture corresponds to what can happen transverse to a component of Σ_T , the second to a component of Σ_R.

(2.5.10) Proposition : Let $\mathcal{V} \longrightarrow S$ be a flat deformation of the curve $V_0 = L_r^r$. Then an improvement $\mathcal{W}$ of $\mathcal{V}$ contains only partition singularities of type $\pi = (1,1, \ldots ,1)$. Sing($\mathcal{V}$) is the union of a number of smooth components mapping 1-1 to S. The map Sing($\mathcal{W}$) $\longrightarrow$ S is **unbranched.**

Proof : The proof will be given in chapter 4 (see (4.1.12)), after we have developed the theory of the fundamental cycle for improvements (§ 3.4). Philosophically the reason is that only the partition singularity of type $\pi = (1,1,\ldots,1)$ has a general hyperplane section of type L_r^r. ∎

Corollary (2.5.4) gives us the possibility to compare $p_g(X) = p_g(X,p)$ with $p_g(X_t) := \Sigma_{q \in X_t} p_g(X_t,q)$.

One can ask whether $p_g(X) \geq p_g(X_t)$, i.e. whether p_g is an upper semicontinuous invariant , as for isolated singularities (see (2.2.6)).
This turns out to be the case (at least for X WNCM).
p_g can be defined using an improvement, so in order to compare $p_g(X)$ with $p_g(X_t)$ it would be useful to have some kind of improvement $\mathcal{Y} \longrightarrow \mathcal{X}$ of the total space of our deformation $\mathcal{X} \longrightarrow S$, which induces an improvement $Y_t \longrightarrow X_t$ for the general fibre, to run the proof parallel to Elkik's. I have been unable to construct such an improvement (say in the sense of (1.4.2)) for $\mathcal{X}$ in general. But actually, because the proof is largely "formal" , we do not really need the space $\mathcal{Y}$, but only the cohomology of certain sheaves on it.

(2.5.11) Let (X,Σ,p) be a WNCM-surface germ.(we assume the CM condition to avoid some technical problems).

Let $\mathcal{X} \xrightarrow{f} S$ be a flat deformation of X over a smooth curve germ (S,0). So $\mathcal{X}$ is also CM. Inside $\mathcal{X}$ we have the surface T of non-normal points of $\mathcal{X}$. Let $\tilde{\mathcal{X}} \xrightarrow{n} \mathcal{X}$ be the normalization map and define $\tilde{T} = n^{-1}(T)$. We denote the map $\tilde{T} \xrightarrow{m} T$ by m. It is a finite branched covering. Let $B \subset T$ be the branch locus of m. Note that by (2.5.10) we have $B \cap X \subseteq \{p\}$.
Let $\Delta \xrightarrow{p} T$ be an embedded resolution of $B \subset T$. So Δ is smooth, p contracts a normal crossing divisor $K \subset \Delta$ and the strict transform $\tilde{B}$ of B is smooth and transversal to K. Now look at the level curves of the induced map $f \circ p$ on Δ. It is a family of smooth curves, degenerating for t=0 into a curve Δ_0. We give it the analytic structure as fibre of the map $\Delta \longrightarrow S$. The curve Δ_0 consists of a compact, non-reduced part C_0 and a non-compact part N_0. It is important that the structure of N_0 is **reduced** , which follows from (2.5.10). The curve N_0 maps via $f \circ p$ to Σ_T. Above a general point of Σ_T there lie as many points of N_0 as the number of irreducible components of $\mathcal{X}$ at that point.
Now we pull back the branched cover $\tilde{T} \xrightarrow{m} T$ via p over Δ and then normalize. The resulting space we call G and we denote the induced mapping $G \xrightarrow{l} \Delta$ by l. Hence a diagram arises:

$$\begin{tikzcd}
G \arrow[r, "q"] \arrow[d, "l"'] & \tilde{T} \arrow[r] \arrow[d, "m"'] & \tilde{\mathcal{X}} \arrow[d, "n"'] \\
\Delta \arrow[r, "p"] & T \arrow[r] & \mathcal{X}
\end{tikzcd}$$

Note that the map $l : G \longrightarrow \Delta$ branches over $\tilde{B} \cup K$ and G has only *cyclic quotient singularities* (see [Lau 1])

(2.5.12) Remark : In general one cannot obtain a smooth G by blowing up Δ further and taking the normalized pull-back. As an example one can take :

$$G = \{ (x,y,z) \in \mathbb{C}^3 \mid z^4 = x.y \} \longrightarrow \Delta = \{(x,y) \in \mathbb{C}^2\}$$

This curious phenomenon does not occur for two or three fold covers.

(2.5.13) Now we look at the surface $\tilde{T} \subset \tilde{\mathcal{X}}$. Take an embedded resolution $\tilde{\mathcal{Y}} \xrightarrow{\pi} \tilde{\mathcal{X}}$ of $\tilde{T}$ in $\tilde{\mathcal{X}}$ and let $\tilde{\Delta}$ be the strict transform of $\tilde{T}$. The exceptional set of π contains three kinds of components.

1) A compact part E, mapping to the point p.
2) A part F_1 mapping properly to $n^{-1}(\Sigma)$.
3) A part F_2 mapping properly to Ξ_h.

We assume E, F_1, F_2 all to be normal crossing divisors, intersecting transversally, and that they are also transversal to $\tilde{\Delta}$. By blowing up further we can even arrange that $\tilde{\Delta} \cap F_2 = \emptyset$.
On $\tilde{\Delta}$ there is an induced family of curves, obtained as fibres of the map $f \circ n \circ \pi : \tilde{\Delta} \longrightarrow S$. The general fibre is smooth, and degenerates into the a curve $\tilde{\Delta}_0$, the fibre over 0 of this family.
We take the analytic structure on $\tilde{\Delta}_0$ as fibre in this family.
$\tilde{\Delta}_0$ consists of a compact part $\tilde{C}_0 = \tilde{\Delta} \cap E$ and a reduced non-compact part $\tilde{N}_0$.
We can consider $\tilde{\Delta} \longrightarrow \tilde{T}$ as a resolution of $\tilde{T}$. But as in (2.5.11) there is also a map $G \longrightarrow \tilde{T}$ which is, up to the quotient singularities of G, also a resolution of $\tilde{T}$. So we can find a space H dominating both $\tilde{\Delta}$ and G.
We put all these spaces in one big diagram:

(2.5.14) **Improvement diagram :**

$$\begin{tikzcd}
H \arrow[r, "r"] \arrow[d, "\sigma"] & \tilde{\Delta} \arrow[r] \arrow[d, "\tau"] & \tilde{\mathcal{Y}} \arrow[d, "\pi"] \\
G \arrow[r, "q"] \arrow[d, "l"] & \tilde{T} \arrow[r] \arrow[d, "m"] & \tilde{\mathcal{X}} \arrow[d, "n"] \\
\Delta \arrow[r, "p"] & T \arrow[r] & \mathcal{X}
\end{tikzcd}$$

The maps p, q, r; σ, τ, π are birational. The maps l,m,n are finite. H , $\tilde{\Delta}$, Δ, $\mathcal{Y}$ are smooth; G has cyclic quotient singularities. All these spaces can considered over S. We do not give names to the inclusion maps.
The ideal situation would be $\tilde{\Delta} = G$, because then we could form the push out on the maps $\tilde{\Delta} \longrightarrow \tilde{\mathcal{Y}}$ and $\tilde{\Delta} \longrightarrow \Delta$ to get a space $\mathcal{Y}$ which induces an improvement of the general fibre X_t.
Although we cannot arrange that $\tilde{\Delta} = G$ in general, they are equal on a certain cohomological level, as one has:

$$Rr_*\mathcal{O}_H \approx \mathcal{O}_{\tilde{\Delta}}$$
$$R\sigma_*\mathcal{O}_H \approx \mathcal{O}_G$$

because G has only cyclic quotient singularities, which are rational. Instead of bothering about glueing, we will construct a sheaf Ω on $\mathcal{X}$, which plays the role of $\pi_*\omega_{\mathcal{Y}}$ of an improvement.
As $\mathcal{Y}$ should be a push out, we should try to do something like (2.4.1).

We start with the exact sequence:

$$0 \longrightarrow \omega_{\tilde{\mathcal{Y}}} \longrightarrow \omega_{\tilde{\mathcal{Y}}}(\tilde{\Delta}) \longrightarrow \omega_{\tilde{\Delta}} \longrightarrow 0$$

of (2.4.4).
By Grauert-Riemenschneider (2.2.3) for $\tilde{\mathcal{Y}}$, this sequense remains exact after applying $(n\circ\pi)_*$ to it.

Further we have: $\sigma_*\omega_H = \omega_G$, $r_*\omega_H = \omega_{\tilde\Delta}$, so also $\tau_*\omega_{\tilde\Delta} = q_*\omega_G$. The map $G \longrightarrow \Delta$ induces a split surjection $1_*\omega_G \longrightarrow \omega_\Delta$ (because G is CM and Δ is smooth). By Grauert-Riemenschneider for the maps p and q it follows that $(p\circ 1)_*\omega_G \longrightarrow p_*\omega_\Delta$ is still surjective. As $(p\circ 1)_*\omega_G = (m\circ q)_*\omega_G = (m\circ\tau)_*\omega_{\tilde\Delta}$ we end up with a surjective map:

$$(m\circ\tau)_*\omega_{\tilde\Delta} \longrightarrow p_*\omega_\Delta$$

(2.5.15) **Definition :** $\Omega := \ker((n\circ\pi)_*\omega_{\tilde{\mathcal{Y}}}(\tilde\Delta) \longrightarrow p_*\omega_\Delta)$
So this is a certain coherent sheaf on
$\mathcal{X}$. It fits in the following diagram with exact rows:

(2.5.16)

$$\begin{tikzcd}
0 \arrow[r] & (n\circ\pi)_*\omega_{\tilde{\mathcal{Y}}} \arrow[r] & (n\circ\pi)_*\omega_{\tilde{\mathcal{Y}}}(\tilde\Delta) \arrow[r] \arrow[d, "\wr"] & (m\circ\tau)_*\omega_{\tilde\Delta} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega \arrow[u, hook] \arrow[r] & (n\circ\pi)_*\omega_{\tilde{\mathcal{Y}}}(\tilde\Delta) \arrow[r] & p_*\omega_\Delta \arrow[r] & 0
\end{tikzcd}$$

and into an exact sequence:

(2.5.17)

$$0 \longrightarrow (n\circ\pi)_*\omega_{\tilde{\mathcal{Y}}} \longrightarrow \Omega \longrightarrow (m\circ\tau)_*\omega_{\tilde\Delta} \longrightarrow p_*\omega_\Delta \longrightarrow 0$$

These are the pendants of (2.4.2) and (2.4.1) respectively.

(2.5.18) All spaces in the improvement diagram map to $\mathcal{X}$. Put $\mathcal{X}^* = \mathcal{X} - \{p\}$ and denote by an upper star above a space in diagram (2.5.14) the inverse image of $\mathcal{X}^*$ in these spaces. For example $\Delta^* = \Delta - K$. Now $H^* \longrightarrow \tilde\Delta^*$ and $H^* \longrightarrow G^*$ so we get a finite map $\tilde\Delta^* \longrightarrow \Delta$. So we can form the push-out $\mathcal{Y}^*$ (see § 1.2) together with a map to $\mathcal{X}^*$. By abuse of notation, we denote this map also by $\pi:\mathcal{Y}^* \longrightarrow \mathcal{X}^*$. We can form the sheaf $\pi_*\omega_{\mathcal{Y}^*}$ on $\mathcal{X}^*$. One can extend this to the sheaf $j_*\pi_*\omega_{\mathcal{Y}^*}$ on $\mathcal{X}$, where j is the inclusion map of $\mathcal{X}^*$ in $\mathcal{X}$. As we assumed $\mathcal{X}$ to be CM, we have that $\omega_{\mathcal{X}} = j_*j^*\omega_{\mathcal{X}}$ so we get an inclusion $j_*\pi_*\omega_{\mathcal{Y}^*} \longrightarrow \omega_{\mathcal{X}}$.

(2.5.19) Lemma : Ω is a coherent, torsion free $\mathcal{O}_{\mathcal{X}}$ - module, mapping into $\omega_{\mathcal{X}}$. Its restriction to $\mathcal{X}^*$ coincides with the sheaf $\pi_*\omega_{\mathcal{Y}^*}$.

proof : Ω is coherent as it is defined as the kernel of a sheaf map between two coherent sheaves. It is torsion free, because $(n\circ\pi)_*\omega_{\tilde{\mathcal{Y}}}$ and $(m\circ\tau)_*\omega_{\tilde{\Delta}}$ are torsion free. Let j as above the inclusion map. So we have $\Omega \longrightarrow j_*j^*\Omega$. The restriction of Ω to $\mathcal{X}^*$ coincides with $\pi_*\omega_{\mathcal{Y}^*}$ because by (2.4.1) and (2.4.2) it sits in the same diagram as the sheaf $j^*\Omega$ by j^*(2.5.16) and j^*(2.5.17). The inclusion $\Omega \longrightarrow \omega_{\mathcal{X}}$ is the composition

$$\Omega \longrightarrow j_*j^*\Omega \longrightarrow j_*\pi_*\omega_{\mathcal{Y}^*} \longrightarrow j_*j^*\omega_{\mathcal{X}} = \omega_{\mathcal{X}}$$

■

(2.5.20) Remark : The case of a smoothing of a normal surface singularity X shows that the sheaves Ω and $j_*\pi_*\omega_{\mathcal{Y}^*}$ in general are really different. The point is that Ω is expected to have only depth 1.

(2.5.21) Corollary : The quotient sheaf $\omega_{\mathcal{X}}/\Omega$ has as support a curve that maps finitely to S. This is because for a point $q \in X - \{p\}$ one has $(\pi_*\omega_{\mathcal{Y}^*})_q \approx \omega_{\mathcal{X},q}$, as by (2.5.7) the transversal singularities are weakly rational.

(2.5.22) Consider now the action of multiplication by t, the local parameter on S, on the exact sequence (2.5.17). There arises a diagram with exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r] & (n\circ\pi)_*\omega_{\tilde{\mathcal{Y}}} \arrow[r] \arrow[d, hook, "t"'] & \Omega \arrow[r] \arrow[d, hook, "t"'] & (m\circ\tau)_*\omega_{\tilde{\Delta}} \arrow[r] \arrow[d, hook, "t"'] & p_*\omega_{\Delta} \arrow[r] \arrow[d, hook, "t"'] & 0 \\
0 \arrow[r] & (n\circ\pi)_*\omega_{\tilde{\mathcal{Y}}} \arrow[r] \arrow[d] & \Omega \arrow[r] \arrow[d] & (m\circ\tau)_*\omega_{\tilde{\Delta}} \arrow[r] \arrow[d] & p_*\omega_{\Delta} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (n\circ\pi)_*\omega_{\tilde{Y}} \arrow[r] & \Omega_0 \arrow[r] & (m\circ\tau)_*\omega_{\tilde{\Delta}_0} \arrow[r] & p_*\omega_{\Delta_0} \arrow[r] & 0
\end{tikzcd}$$

The first, third and fourth column are obtained from sequences like $0 \longrightarrow \omega_{\tilde{\mathcal{Y}}} \xrightarrow{t} \omega_{\tilde{\mathcal{Y}}} \longrightarrow \omega_{\tilde{Y}} \longrightarrow 0$, which remain exact under direct image by Grauert-Riemenschneider (2.2.3). Remember that $\tilde{Y}$ is the zero fibre of $\tilde{\mathcal{Y}} \longrightarrow S$. (Strictly speaking we should work with $\omega_{\tilde{\mathcal{Y}}/S}$ instead of $\omega_{\tilde{\mathcal{Y}}}$).
The sheaf Ω_0 is defined by the exactness of the second column. The content of the diagram is then the exactness of the bottom row.

(2.5.23) Look at the sequence :

$$0 \longrightarrow \omega_{\mathcal{X}} \xrightarrow{t} \omega_{\mathcal{X}} \longrightarrow \omega_X \longrightarrow 0$$

It is exact because $\mathcal{X}$ is CM.
There is a mapping from the second column of (2.5.22) to this sequence, induced by the inclusion of (2.5.19).
Now we can apply the index lemma (2.1.10) to get:
<u>conclusion</u> : $p_g(X_t) = \mathrm{Index}(\ \Omega_0 \longrightarrow \omega_X)$
with X_t the general fibre of the family.

(2.5.24) We have to study very carefully what happens in the zero-fibre of f in all spaces of diagram (2.5.14). Let us first remind how the zero fibre $\tilde{Y}$ in $\tilde{\mathcal{Y}}$ looks like. It consists of three kinds of divisors:
1) $\tilde{X}$, the strict transform of X.
2) the components E, mapping to the point p.
3) the components F_1, mapping properly to Σ.
So : $\tilde{Y} = \tilde{X} \cup E \cup F_1$.
Further there is the divisor $\tilde{\Delta}$ to worry about.
We put $D := E \cap (\tilde{X} \cup F_1)$.
Remember the curve $\tilde{\Delta}_0 = \tilde{\Delta} \cap \tilde{Y}$ of (2.5.13). It consists of a non-compact part $\tilde{N}_0 = \tilde{\Delta} \cap (\tilde{X} \cup F_1)$ and a compact part $\tilde{C}_0 = \tilde{\Delta} \cap E$.
Via the mappings $H \xrightarrow{r} \tilde{\Delta}$ and $H \xrightarrow{\sigma} G \xrightarrow{l} \Delta$ of diagram (2.5.14), these components $\tilde{N}_0$ correspond to the non-compact components N_0 of the curve Δ_0 . This provides us with a finite map $\tilde{N}_0 \longrightarrow N_0$, which is branched possibly over the set $P := N_0 \cap C_0$. The inverse image of P is $\tilde{P} = \tilde{N}_0 \cap \tilde{C}_0$.
We can form the push out space Z on the maps $\tilde{N}_0 \longrightarrow N_0$ and $\tilde{N}_0 \longrightarrow \tilde{X} \cup F_1$ as in § 1.2. As $\tilde{N}_0$ and N_0 are reduced Z will be weakly normal (c.f. (1.2.20)). By the universal property we obtain

a map $Z \longrightarrow X$, which we call by abuse simply π. Above a general point $q \in \Sigma$ we have the improvement of the transversal singularity, intersected with the zero fibre.

The inclusions $\tilde{X} \cup F_1 \longrightarrow \tilde{Y}$, $\tilde{N}_0 \longrightarrow \tilde{\Delta}_0$ and $N_0 \longrightarrow \Delta_0$ induce another big diagram with exact rows and columns:

(2.5.25) **Diagram :**

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (n\circ\pi)_*\omega_{\tilde{X}\cup F_1} \arrow[r] \arrow[d] & \pi_*\omega_Z \arrow[r] \arrow[d] & (m\circ\tau)_*\omega_{\tilde{N}_0} \arrow[r] \arrow[d] & p_*\omega_{N_0} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (n\circ\pi)_*\omega_{\tilde{Y}} \arrow[r] \arrow[d] & \Omega_0 \arrow[r] \arrow[d] & (m\circ\tau)_*\omega_{\tilde{\Delta}_0} \arrow[r] \arrow[d] & p_*\omega_{\Delta_0} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (n\circ\pi)_*\omega_E(D) \arrow[r] \arrow[d] & \mathfrak{M} \arrow[r] \arrow[d] & (m\circ\tau)_*\omega_{\tilde{\Delta}_0}(\tilde{P}) \arrow[r] \arrow[d] & p_*\omega_{C_0}(P) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

The first, third and fourth column arise from the above mentioned inclusion maps. The top row is the push-out sequence (2.4.1), to which a direct image is applied. Again the Grauert-Riemenschneider theorem garantees that exactness is preserved. The middle row is the bottom row of diagram (2.5.22). Using the slightly extended diagrams (2.4.2) for ω_Z and $(2.5.16)_0$ for Ω_0 one can check that there really is a sheaf map $\pi_*\omega_Z \longrightarrow \Omega_0$.
The content of the diagram is that the cokernel $\mathfrak{M}$ of this last map fits in the exact bottom row.

(2.5.26) **Corollary :** The dimension $\dim(H^0(\mathfrak{M}))$ is equal to

$\dim H^0(\omega_E(D)) + \dim H^0(\omega_{\tilde{C}_0}(\tilde{P})) - \dim H^0(\omega_{C_0}(P))$

where the global sections are taken over the spaces on which these sheaves live.

(2.5.27) **Lemma :** Let $Y \xrightarrow{\rho} X$ be an improvement of X. Then $\rho_*\omega_Y = \pi_*\omega_Z$.

proof : Remember that we had a map $Z \xrightarrow{\pi} X$. It maps N_0 to Σ and as N_0 is smooth, the map $N_0 \longrightarrow \Sigma$ factors over the normalization $A \longrightarrow \Sigma$ (we ran out of symbols...). Let U be the image of $\tilde{X}$ in in Z and F the image of F_1 and form the push-out space Y on the maps $N_0 \longrightarrow U$, $N_0 \longrightarrow A$. The map $Z \xrightarrow{\pi} X$ factorizes into $Z \xrightarrow{c} Y \xrightarrow{\rho} X$.

Claim 1) $Y \xrightarrow{\rho} X$ is an improvement of X.

Claim 2) $c_*\omega_Z = \omega_Y$.

1) : The normalization of Y is $\tilde{X}$, so is smooth. The singular locus of Y is the curve A , the inverse image of A in $\tilde{X}$ is the curve $\tilde{N}_0$, so is also smooth. Y is weakly normal, and outside the point p the spaces X and Y are isomorphic, by construction.

2) : Because the transverse singularities of X are weakly rational the fibres of the map $Z \longrightarrow Y$ above a point $q \in \Sigma - \{p\}$ do not carry higher cohomology. From this one can deduce that $Rc_*\mathcal{O}_Z \approx \mathcal{O}_Y$, and by duality $c_*\omega_Z = \omega_Y$.
From these two facts the lemma follows. ∎

Now we can finish the proof of the semicontinuity theorem:

(2.5.28) **Theorem :** Let (X,Σ,p) be a WNCM-surface germ. Let $\mathcal{X} \xrightarrow{f} S$ be a flat deformation over a smooth curve germ S, and let $X_s = f^{-1}(s)$ be the fibre over s.
Then :

$$p_g(X) = p_g(X_s) + \dim H^0(\mathfrak{M})$$

so in particular:

$$p_g(X) \geq p_g(X_s)$$

Here $\dim H^0(\mathfrak{M})$ is the number given by (2.5.26).

proof : Consider the diagram of maps:

$$\begin{array}{ccc} \Omega_0 & \longrightarrow & \omega_X \\ \uparrow & & \uparrow \\ \pi_*\omega_Z & = & \rho_*\omega_Y \end{array}$$

From the additivity of the index we get :

Index$(\Omega_0 \longrightarrow \omega_X)$ + dim Coker $(\pi_*\omega_Z \longrightarrow \Omega_0)$ =

= dim Coker $(\rho_*\omega_Y \longrightarrow \omega_X)$, where we applied lemma (2.5.27) by identifying $\pi_*\omega_Z$ with $\rho_*\omega_Y$. By the conclusion of (2.5.23) and formulas (2.5.26) and (2.4.6) the theorem follows ■

(2.5.29) Remark : The whole set-up of this proof may look cumbersome and would even look more so when we did not assume X to be CM, which allowed us to work with a single dualizing sheaf, instead of a complex. Probably a more systematic use of simplicial resolutions would both clarify and generalize the above proof. This is an interesting future project.

(2.5.30) A slightly different approach to the semicontinuity of p_g is as follows:

Start with a flat deformation $\mathcal{X} \xrightarrow{f} S$. Now first normalize $\mathcal{X}$ to $\tilde{\mathcal{X}}$ and study the resulting family $\tilde{\mathcal{X}} \longrightarrow S$. As $\tilde{\mathcal{X}}$ is normal, this is a flat deformation of its zero fibre X_1 to isolated singularities (type 2) under (2.5.5)). It is easy to see that X_1 is an AWN-surface. For such a deformation to isolated singularities it is easy to prove (2.5.26) and (2.5.27) : no glueing with a Δ is needed. This leads to:

$$p_g(X_1) = \dim H^0(\omega_E(D)) + p_g(\tilde{X}_s)$$

where $\tilde{X}_s$ is the normalization of the general fibre X_s.

Let $\Sigma_1 = \mathrm{Sing}(X_1)$ and $\tilde{\Sigma}_1$ the inverse image under the normalization map $\tilde{X} \longrightarrow X_1$. Now the formula of (2.5.28) is equivalent to :

$$\delta(\tilde{\Sigma}) - \delta(\Sigma) = \delta(\tilde{\Sigma}_s) - \delta(\Sigma_s) + \delta(\tilde{\Sigma}_1) - \delta(\Sigma_1) + h^0(\omega_{\tilde{C}}(\tilde{P})) - h^0(\omega_C(P))$$

These last two terms are also δ - invariants of certain curves. Let $\tilde{U} \longrightarrow \tilde{T}$, $U \longrightarrow T$ be the normalizations of the surfaces $\tilde{T}$ and T ; $\tilde{U}_0$ and U_0 the zero fibres of the induced maps to S. Then :

$$\delta(U_0) = h^0(\omega_C(P))$$

$$\delta(\tilde{U}_o) = h^0(\omega_{\tilde{C}}(\tilde{P}))$$

In the *special case* that the surfaces $\tilde{T}$ and T are non-singular at a generic point of their zero fibres $\tilde{T}_0$ and T_0 one can use (2.1.7) to write :

$$\delta(\tilde{T}_0) = \delta(\tilde{\Sigma}_s) + \delta(\tilde{U}_0)$$

$$\delta(T_0) = \delta(\Sigma_s) + \delta(U_0)$$

Hence, in that special case:

$$\delta(\tilde{\Sigma}) - \delta(\Sigma) = \delta(\tilde{\Sigma}_1) - \delta(\Sigma_1) + \delta(\tilde{T}_0) - \delta(T_0)$$

This looks really simple, but I have been unable to establish this relation directly.

(2.5.31) Remark : From the above discussion it follows that (at least when X is WNCM) the number $\delta(\tilde{\Sigma})-\delta(\Sigma)$ is also upper semicontinuous. Of course the invariant "p_g of the normalization" is not semicontinuous, nor are $\delta(\tilde{\Sigma})$ and $\delta(\Sigma)$.

(2.5.32) We conclude with a remark on the case of a *smoothing* $\mathcal{X} \xrightarrow{f} S$ (type 1) under (2.5.5)). When we specialize the formula of (2.5.28) to this case we obtain:

$$p_g(X) = \dim H^0(\omega_E(D))$$

Comparing this with (2.2.7) we can interprete the geometric genus in this more general situation of a smoothing of a WNCM surface germ still as the holomorphic part of the vanishing cohomology:

$$p_g(X) = Gr_F^2 H^2(R\Phi)$$

In my opinion this interpretation is the 'deeper reason' for the semicontinuity of p_g.

CHAPTER 3

CYCLES ON IMPROVEMENTS

For normal surface singularities there exists a well developed calculus of one-cycles on the resolution. These cycles may be thought of as infinitesimal neighbourhoods of the exceptional locus in the resolving surface. In this way one can, at least for the simplest classes of singularities, like rational and minimally elliptic singularities (see Chapter 4), reduce the study of the singular surface to the study of a non-reduced curve: the fundamental cycle.

In this chapter we want to set up a similar theory for improvements of weakly normal surfaces. Although the generalization is straightforward, the distinction between Weil and Cartier divisors gives some unexpected twists.

§ 3.1 Cycles and subspaces

We fix the usual situation as in (1.4.8). We assume that the singular locus Δ of the improvement space Y is transverse to the exceptional locus E.

Remember (see(1.3.5)) the special lines L_i , $i=1,2,\ldots,k$ on the partition singularity X_π. For every special point $s \in S = \Delta \cap E$ we can choose an isomorphism $(Y,s) \longrightarrow (X_{\pi(s)},0)$ such that $(E,s) \longrightarrow (L,0)$, where $L = \cup L_i$,and where the partition $\pi(s) = (\alpha(1,s),\alpha(2,s),\ldots,\alpha(k(s),s))$ depends on the special point s under consideration.

Let us briefly recall the distinction between *Weil* and *Cartier* divisors on a space Z. A Weil divisor D on Z is a finite sum $D = \sum a_i Y_i$, with the $a_i \in \mathbb{Z}$ and the Y_i irreducible subspaces of Z of pure codimension 1. It is called *effective* if $a_i \geq 0$ for all i; its *support*, supp(D), is the set $\cup \{Y_i | a_i \neq 0\}$. D is called a *Cartier divisor* at $x \in Z$ if around x, D is the divisor of a meromorphic function. A Weil divisor is a Cartier divisor if it is Cartier at every point.(see[Ha 2] for more precise definitions).

The set of Weil divisors on a space Z forms in an obvious way a group and the set of Cartier divisors is a subgroup.

(3.1.1) **Lemma & definition :** The group W of Weil divisors on X_π with support in the set L is

$$W = \oplus_{i=1}^{k} \mathbb{Z}.L$$

The subgroup of Cartier divisors with support in L is

$$C = \mathbb{Z}.\Lambda$$

where $\Lambda := \sum_{i=1}^{k} \alpha(i).L_i$. We call Λ the *fundamental Cartier divisor* on the partition singularity X_π .

Let us return to the situation of an improvement Y.

(3.1.2) **Definition :** A (Weil) *cycle* on Y is a Weil divisor on Y with support contained in E.

We denote the group of cycles by W_Y or W. So we have :

$$W_Y = \oplus \mathbb{Z}.E_i$$

where E_i are the irreducible components of E.

We give W a *partial ordering* as follows :

if $A = \sum a_i.E_i$, $B = \sum b_i.E_i$, then we write $A \geq B$ iff $a_i \geq b_i$ for all i . We take $A > B$ to mean $A \geq B$ and $A \neq B$.

The set $W^{\geq 0} = \{ A \in W \mid A \geq 0 \}$ is called the *effective cone* or the set of *positive cycles.* Similarly we use the notation $W^{>0}$.

Inside W there is the subgroup C of *Cartier cycles.*

We put $C^{\geq 0} = W^{\geq 0} \cap C$ etcetera.

(3.1.3) **Remark :** It can happen that the group of Cartier cycles consists of 0 only.

(3.1.4) To every element $A = \sum a_i.E_i \in W^{\geq 0}$ there corresponds a unique *Cohen-Macaulay subspace* of Y, which we denote by $(A,\mathcal{O}_A)$, or simply by A. We denote by $\mathcal{O}_Y(-A)$ the *ideal sheaf* of A. It is by definition the largest ideal such that on a point $a \in E_i - S$ one has $\mathcal{O}_Y(-A)_a = \mathcal{O}_{Y,a}.x^{a_i}$, where $x = 0$ is a local equation for E_i at a. We have : $\mathcal{O}_A = \mathcal{O}_Y/\mathcal{O}_Y(-A)$.

For a cycle $A \in W^{\geq 0}$ the following holds :

$$A \in C^{\geq 0} \quad \text{if and only if} \quad \mathcal{O}_Y(-A) \text{ is invertible.}$$

The main difference between cycles on improvements and on resolutions is this distinction between Weil and Cartier cycles. Another difference is that the subspace corresponding to a cycle on an improvement does not have embedding dimension two at the special points S in general. Usually they are not even Gorenstein at these points.

§ 3.2 Cycles and modifications.

We discuss the behaviour of cycles with respect to normalization and blowing up points.

(3.2.1) Behaviour under normalization. Let $\tilde{Y} \xrightarrow{n} Y$ be the normalization map.

We put as usual $\tilde{E}_i := n^{-1}(E_i)$. This sets up a one to one correspondence between irreducible components of $\tilde{E}$ and E and hence gives rise to isomorphisms:

$$n_* : W_{\tilde{Y}} \longrightarrow W_Y \qquad \text{and} \qquad n^* : W_Y \longrightarrow W_{\tilde{Y}}$$

We will usually write $\tilde{A}$ for $n^*(A)$.

These isomorphisms make it possible to confuse cycles on Y and $\tilde{Y}$. We will as much as possible distinguish between these two notions, as the associated subspaces in Y and $\tilde{Y}$ are very different.

To compare these associated subspaces A and $\tilde{A}$ we use the exact sequence:

(3.2.2)

$$0 \longrightarrow \mathcal{O}_A \longrightarrow n_*\mathcal{O}_{\tilde{A}} \longrightarrow \mathcal{G}_A \longrightarrow 0$$

Here $\mathcal{G}_A$ is concentrated on the set S and its length is an invariant of A.

(3.2.3) Definition : $\gamma_A(s) := \dim_{\mathbb{C}} \mathcal{G}_{A,s}$

$$\gamma_A := \Sigma_{s\in S}\, \gamma_A(s)$$

So we can compare the Euler characteristic of A and $\tilde{A}$:

(3.2.4)

$$\chi(\mathcal{O}_{\tilde{A}}) = \chi(\mathcal{O}_A) + \gamma_A$$

The Euler characteristic on the normalization $\chi(\mathcal{O}_{\tilde{A}})$ can be computed using the adjunction formula on $\tilde{Y}$ (see § 3.3).
This local invariant $\gamma_A(s)$ can be computed on the partition singularity, using the isomorphism $(Y,s) \approx (X_{\pi(s)},0)$:

(3.2.5) Proposition : Let X_π , $\pi = (\alpha(1),\alpha(2),\ldots,\alpha(k))$ be a partition singularity.
Let $A = \Sigma_{i=1}^{k} a_i.L_i$ be a positive cycle. Then :

$$\gamma_A(0) = \Sigma_{i=1}^{k} a_i \quad - \quad \min\,(\lceil a_i/\alpha(i)\rceil,\ i=1,2,\ldots,k)$$

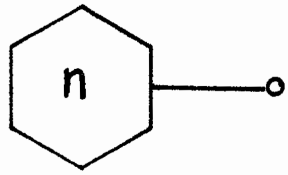
where $\lceil x\rceil$ means rounding up.

Proof : We use the coordinates (1.3.1). The ideal of $\tilde{A}$ is $\mathcal{J}_{\tilde{A}} = u.\mathcal{O}_{\tilde{X}}$, where $u = (u^{a_1},u^{a_2},\ldots,u^{a_k})$. Then we have
$\gamma_A(0) = \dim\,(\mathcal{O}_{\tilde{X}}/(\mathcal{O}_X + \mathcal{J}_{\tilde{A}})) = \dim\,(\mathcal{O}_{\tilde{\Sigma}}/(\mathcal{O}_\Sigma + u.\mathcal{O}_{\tilde{\Sigma}}))$.
The space $\mathcal{O}_{\tilde{\Sigma}}/u.\mathcal{O}_{\tilde{\Sigma}}$ has dimension $\Sigma\, a_i$. The number of powers of the element X (the coordinate on Σ, see (1.3.4)) we have to divide away, make up the second part of the formula. ∎

(3.2.6) Remark : We note the following two special cases of this formula:

1) If A is reduced (a_i = 0 or 1) then $\gamma_A(0) = \Sigma\, a_i - 1$.
   Note that $\Sigma\, a_i$ is the number of irreducible components of A at the special point.
2) If A is Cartier, so $A = m.\Lambda$, then $\gamma_A(0) = m.(\alpha - 1)$.

(3.2.7) Example : Let Y be an improvement with the following improvement graph:



Let $A = a.F$, where F is the (-1)-curve.
Then $\chi(\mathcal{O}_A) = \chi(\mathcal{O}_{\tilde{A}}) - \gamma_A = a.(a+1)/2 - (a - \lceil (a/n) \rceil)$.
So this is, due to the non-linearity of $A \longrightarrow \gamma_A$, not a polynomial in a. But for A Cartier, $a = m.n$, it is a polynomial in m.

(3.2.8) Behaviour under blowing up : There are several cases to distinguish. Let us first consider the case of blowing up in a smooth point of Y.

(3.2.9) Let $Z \xrightarrow{\rho} Y$ the blowing up map. It introduces an extra (-1)-curve on Z and induces transformations:

$$\rho_* : W_Z \longrightarrow W_Y \quad ; \quad C_Z \longrightarrow C_Y$$

$$\rho_* : W_Y \longrightarrow W_Z \quad ; \quad C_Y \longrightarrow C_Z$$

defined in the usual way (see [Ha 2], [B-P-V]). Here ρ^* denotes the *total* transform. If $A \in W_Y$, we denote by $\bar{A} \in W_Z$ its *strict* transform.

(3.2.10) More interesting is the situation at the special points. For point $s \in S$ we define modifications $\varepsilon_i(s)$, $i = 1,2,\ldots,k(s)$, where $k(s)$ is as usual the number of irreducible components of (Y,s). Using the isomorphism $(Y,s) \approx (X_{\pi(s)},s)$, it suffices to describe it for a partition singularity X_π .

(3.2.11) Definition : Let X_π , $\pi = (\alpha(1),\alpha(2),\ldots,\alpha(k))$ be a partition singularity. Let Σ its singular locus. The *i-th elementary modification* ε_i

$$\varepsilon_i : X_{\pi,i} \longrightarrow X_\pi$$

is constructed as follows:

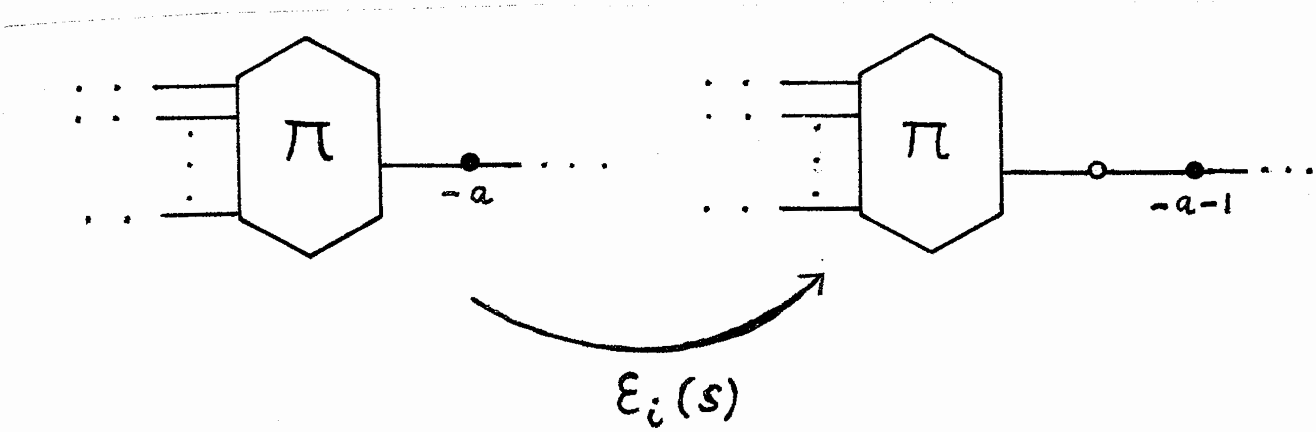
1) Normalize X_π to get $\tilde{X} = \coprod_{i=1}^{k} \tilde{X}_i$.
2) Blow up in the i-th component of $\tilde{X}$ and get $\tilde{Y} \longrightarrow \tilde{X}$.
3) Consider the strict transform $\bar{\Sigma}$ of $\tilde{\Sigma}$. So we have maps $\bar{\Sigma} \longrightarrow \tilde{Y}$ and $\bar{\Sigma} \longrightarrow \Sigma$ (via $\bar{\Sigma} \longrightarrow \tilde{\Sigma}$)
4) Form the push out space $X_{\pi,i}$ on the two maps under 3). By the universal property we get a map $\varepsilon_i : X_{\pi,i} \longrightarrow X_\pi$.

(3.2.12) **Remark :** The elementary transformation ε_i is the same as blowing up X_π in the ideal

$$I_i = (X, \{ \Pi_{j=1}^{r} Y_{i,\beta(j)} | \Sigma_{j=1}^{k}\beta(j) = \alpha(i) - r\}_{r=1}^{\alpha(k)})$$

but this description is not of much help.
When we perform such an elementary modification $\varepsilon_i(s)$ we introduce one new curve on the improvement. It is of type 1 in the terminology of (1.4.10). We call this curve $F_i(s)$.
On the level of improvement graphs $\varepsilon_i(s)$ induces a transformation that looks as follows:

(3.2.13) **Elementary modification :**



The modification $\varepsilon_i(s) : Z \longrightarrow Y$ induces transformations:

$$\varepsilon_i(s)^* : W_Y \longrightarrow W_Z \quad ; \quad C_Y \longrightarrow C_Z$$

$$\varepsilon_i(s)_* : W_Z \longrightarrow W_Y$$

Note that there is **no** natural map $C_Z \longrightarrow C_Y$.

(3.2.14) Definition : A Cartier model Z is an improvement obtained from another improvement Y by performing at all special points $s \in S$ of Y the transformation

$$\varepsilon(s) := \varepsilon_1(s)\circ\varepsilon_2(s)\circ\ldots\circ\varepsilon_{k(S)}$$

We call $\varepsilon := \prod \varepsilon(s) : Z \longrightarrow Y$ the Cartier modification.
Let $s \in S$ be a special point of Z (we can safely identify the set of special points of Y and Z). In the neighbourhood of s on Z there now exists a Cartier divisor

$$F(s) = \Sigma_{i=1}^{k} \alpha(i,s).F_i(s)$$

It corresponds via the isomorphism $(Z,s) \approx (X_{\pi(s)},s)$ to the Cartier divisor Λ of (3.1.1).

Cartier models have the convenient property that their exceptional set E is covered by the supports of indecomposable Cartier divisors. These are of two types:

1) The divisors $F(s)$, $s \in S$
2) The irreducible curves E_i that do not intersect Δ.

We can write :

$$W = \oplus\, \mathbb{Z}.F_i(s) \oplus \oplus\, \mathbb{Z}.E_i$$

$$C = \oplus\, \mathbb{Z}.F(s) \oplus \oplus\, \mathbb{Z}.E_i$$

Every Weil divisor A is contained in a unique smallest Cartier divisor $\mathscr{C}(A)$:

(3.2.15) $\quad \mathscr{C}(A) = \inf\{ B \in C \mid B \geq A \}$

Another interesting modification is just blowing up at a special point $s \in S$.

(3.2.16) Proposition : Let X_π , $\pi = (\alpha(1),\alpha(2),\ldots,\alpha(k))$, be a partition singularity.

Let $B(X_\pi) \xrightarrow{b} X_\pi$ be the blowing up in 0. Then :
$b^{-1}(0)$ consists of a $\mathbb{P}^1$ in every irreducible component of $B(X_\pi)$, on the i-th component with multiplicity $\alpha(i)$. They intersect in a unique point q, where $B(X_\pi)$ has a singularity isomorphic to X_π itself. More precisely: $(B(X_\pi),b^{-1}(0),q) \approx (X_\pi,\Lambda,0)$.
Further, on the i-th component of $b^{-1}(0)$ the space $B(X_\pi)$ has an extra $A_{\alpha(i)-1}$ - singularity.

proof : It suffices to analyse what happens to an irreducible component of X_π , so we assume $\pi = (n)$. We write the equations as in (1.3.5):

$$\mathrm{rank}\begin{bmatrix} y_0 & y_1 & \cdots & y_{n-1} \\ y_1 & y_2 & \cdots & x.y_0 \end{bmatrix} \le 1$$

The substitution $x = s$; $y_i = s.t_i$, $i=0,1,\dots,n-1$ transforms the matrix into s times this same matrix with x replaced by s and y_i by t_i, so in this chart we get back the same partition singularity. The substitution $x = s.w$; $y_0 = s$; $y_i = s.t_i$ leads to the equations $t_i = t_1.t_{i-1}$ $i = 2,3,\dots,n-1$ and $s.w = t_1.t_{n-1}$, hence in this chart we get an A_{n-1} - singularity. One can check that in the other charts $B(X_\pi)$ is smooth. It is almost clear that the multiplicities are as stated, because blowing up in 0 is the same as making the maximal invertible, so $b^{-1}(0)$ is Cartier on the blow up $B(X_\pi)$ ∎

(3.2.17) **Definition :** Let Y be an improvement and $s \in S$ a special point. The η - *modification*

$$\eta(s) : Z \longrightarrow Y$$

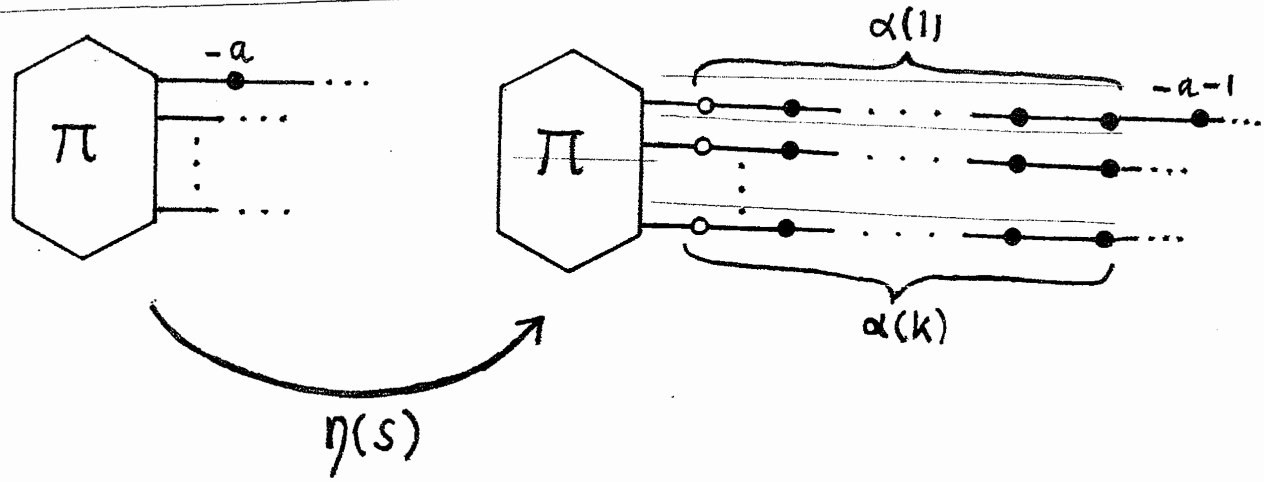
is obtained by first blowing up at s and then taking the minimal resolution of the A_k - singularities that appear.

(3.2.18) **Remark :** $\eta(s)$ can be described in terms of the elementary transformations $\varepsilon_i(s)$ as :

$$\eta(s) = \varepsilon_1(s)^{\alpha(1)} \circ \varepsilon_2(s)^{\alpha(2)} \circ \dots \circ \varepsilon_{k(s)}(s)^{\alpha(k(s))}$$

It has the following effect on the level of improvement graphs:

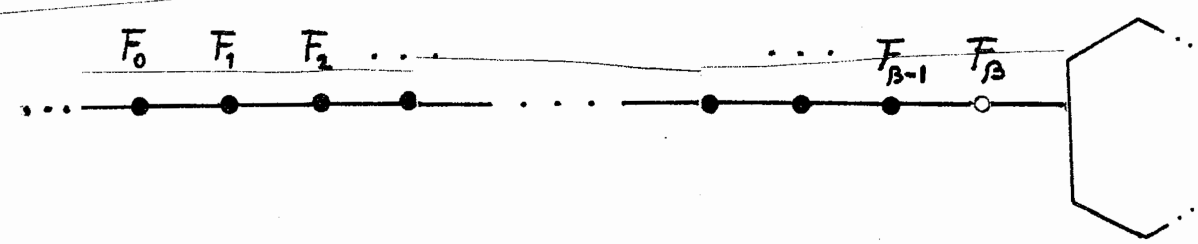
(3.2.19) η - transformation.



(3.2.20) **Definition :** An *elementary chain* is a chain of curves F_i , $i = 0,1,\ldots,\beta$, on an improvement having the following properties:

1) Each curve F_i is a smooth $\mathbb{P}^1$.
2) $F_i \cap S = \emptyset$, $i \neq \beta$; $F_\beta \cap S = s$.
3) The self-intersection of F_i is -2 , $i \neq \beta$ the self-intersection of F_β is -1 .
4) The curves F_i , $i \neq 0$, do not intersect any other irreducible component of the exceptional set E.

We call F_0 the *begin* of the chain and F_β the *end*.
The number β is called the *length* of the chain. A chain *terminates* at the special point s . The number $\alpha(i,s)$ of the partition singularity at which the chain terminates we call the *index*.
So it looks like:



The effect of an elementary modification on an elementary chain is the following: The (strict transform of) F_β becomes a -2 curve, and a new curve $F_{\beta+1}$ is introduced, which now is the new end.

§ 3.3 Intersection Numbers and the Canonical Cycle

On a surface a Weil and a Cartier divisor can be intersected and on a smooth surface the arithmetic genus of a curve can be expressed in intersections with one special divisor: the canonical divisor K. As improvements are not smooth we have to reconsider these questions.

(3.3.1) **Definition :** Let A be a proper curve and $\mathcal{F}$ a sheaf with the property that the rank of $\mathcal{F}$ on every irreducible component is a fixed number $rk(\mathcal{F})$.
The *degree of* $\mathcal{F}$ is the number:

$$deg(\mathcal{F}) := \chi(\mathcal{F}) - rk(\mathcal{F}).\chi(\mathcal{O}_A)$$

As rank and χ are additive over short exact sequences, deg also is. For a line bundle $\mathcal{L}$, $deg(\mathcal{L})$ is just the usual degree (Riemann-Roch). One has $deg(\mathcal{F}\otimes\mathcal{L}) = deg(\mathcal{F}) + deg(\mathcal{L})$ for every sheaf $\mathcal{F}$ as above and every line bundle $\mathcal{L}$ on A (c.f. [B-P-V]).

(3.3.2) **Definition :** Let Z be a surface, A and B curves on Z. Assume that B is Cartier.
The *intersection number* of A and B is the number

$$(A.B) = A.B := deg(\mathcal{O}_Z(B)\otimes\mathcal{O}_A)$$

(3.3.3) **Proposition :** Let $\tilde{Z} \xrightarrow{\rho} Z$ be a modification. Let A and B as above. Then:

$$A.B = \rho^*A.\rho^*B$$

where ρ^* is the total transform.

proof : Well-known, or see [Ha 2].

Let us return to our situation of an improvement Y and consider the normalization $\tilde{Y} \xrightarrow{n} Y$. For $A \in W_Y^{\geq 0}$, $B \in C_Y^{\geq 0}$ we can form with (3.3.2) the intersection number A.B and by linear extension we get an intersection pairing:

$$W_Y \otimes C_Y \longrightarrow \mathbb{Z} \quad ; \; A\otimes B \longmapsto A.B$$

Now by (3.3.3) we have $A.B = \tilde{A}.\tilde{B}$ where $\tilde{A}$ and $\tilde{B}$ are the cycles on $\tilde{Y}$ corresponding to A and B. Hence we can *extend* the above pairing to a pairing between two Weil divisors:

(3.3.4)

$$W_Y \otimes W_Y \longrightarrow \mathbb{Z} \quad ; \; A \otimes B \longmapsto \tilde{A}.\tilde{B} \; (=: A.B)$$

A fundamental fact about this pairing is that it is *negative definite*, by Mumfords result for normal surface singularities (see[Mu]). By restriction one obtains a pairing

$$C_Y \otimes C_Y \longrightarrow \mathbb{Z}$$

which is also negative definite.

We mention all this, because there is another natural extension of the intersection pairing $W^{\geq 0} \otimes C^{\geq 0} \longrightarrow \mathbb{Z}$.

Note the following useful:

(3.3.5) Lemma : Let $A \in W_Y^{\geq 0}$, $B \in C_Y^{\geq 0}$. Then:

$$\chi(\mathcal{O}_{A+B}) = \chi(\mathcal{O}_A) + \chi(\mathcal{O}_B) - A.B$$

proof : This is an easy exercise in diagram chasing and definition reading. □

One can read this last equation as a definition of A.B and so extend the intersection pairing to two Weil divisors. We do not take this approach, because due to the non-linearity of the function $A \longmapsto \gamma_A$ (see (3.2.7)), this would not give a bilinear pairing. Usually (3.3.5) is wrong for two Weil divisors.

(3.3.6) The Canonical Cycle. Let Y be an improvement and let $A \in C_Y^{\geq 0}$. Then because Y is CM and A Cartier there is an exact sequence:

$$0 \longrightarrow \omega_Y \longrightarrow \omega_Y(A) \longrightarrow \omega_A \longrightarrow 0$$

or : $$\omega_A = \omega_Y(A) \otimes \mathcal{O}_A \quad (\text{"Adjunction formula"})$$

(c.f (2.4.4))

From this we get by taking degrees:

$$\deg(\omega_A) = \deg(\omega_Y \otimes \mathcal{O}_A) + \deg(\mathcal{O}_Y(A) \otimes \mathcal{O}_A)$$
$$= \deg(\omega_Y \otimes \mathcal{O}_A) + A^2$$

Now because ω_A is dualizing : $\chi(\omega_A) = -\chi(\mathcal{O}_A)$ so $\deg(\omega_A) = -2.\chi(\mathcal{O}_A)$
Hence:

(3.3.7)

$$\chi(\mathcal{O}_A) = -(1/2).(\ \deg(\omega_Y \otimes \mathcal{O}_A) + A^2\ )$$

(3.3.8) **Notation :** Let Y be an improvement, G and H two divisors on Y, not neccesarily with support $\subseteq$ E. Put:

$$G \underset{\sim}{w} H \quad \text{if} \quad G.A = H.A \quad \text{for all } A \in W_Y$$
$$G \underset{\sim}{c} H \quad \text{if} \quad G.B = H.B \quad \text{for all } B \in C_Y$$

Now let $K_{\tilde{Y}}$ be a canonical divisor on $\tilde{Y}$, so $\omega_{\tilde{Y}} \approx \mathcal{O}_{\tilde{Y}}(K_{\tilde{Y}})$.

(3.3.9) **Proposition :** Let Y be a Cartier model. Then there exists a *unique* cycle $L_Y \in C_Y \otimes \mathbb{Q}$ such that for all $A \in C_Y^{\geq 0}$ the following holds:

$$\deg(\omega_Y \otimes \mathcal{O}_A) = L_A.A$$

Define a $\mathbb{Q}$ - divisor P on $\tilde{Y}$ by:

$$P := \Sigma_{s\in S}\ 2.\left[\ \frac{\alpha(s) - 1}{\alpha(s)}\ \right].\tilde{\Lambda}(s)$$

Then one has: $L_Y \underset{\sim}{c} n_*(\ K_{\tilde{Y}} + P\ )$

proof : By (3.3.7) $\deg(\omega_Y \otimes \mathcal{O}_A) = 2.\chi(\mathcal{O}_A) - A^2$. By the usual adjunction formula on $\tilde{Y}$ one has:
$K_{\tilde{Y}}.\tilde{A} = -2.\chi(\mathcal{O}_{\tilde{A}}) - \tilde{A}^2$. So by (3.2.5) we get :

$$\deg(\omega_Y \otimes \mathcal{O}_A) = K_{\tilde{Y}}.\tilde{A} + 2.\gamma_A$$

As A is Cartier we can write $A = \Sigma_{s\in S}\ \lambda(s).F(s)$ + other terms (c.f.(3.2.14)). Now use (3.2.6) to find:

$$\deg(\omega_Y \otimes \mathcal{O}_A) = (K_{\tilde{Y}} + P).\tilde{A}$$

with P as above. As the intersection pairing on $\tilde{Y}$ is non-degenerate one can find a unique L_Y with the stated properties ∎

(3.3.10) Remark : In general we do not have for $A \in W_Y^{\geq 0}$

$$\deg(\omega_Y \otimes \mathcal{O}_A) = L_Y.\tilde{A}$$

This deg is non-linear in A and is given by a more complicated formula (which will not be our concern here). There is however one notable special case: the case that Y is Gorenstein. By (1.3.7) this only happens when Y has only A_∞ and D_∞ singularities. Let K_Y be a canonical divisor on Y, i.e $\omega_Y \approx \mathcal{O}_Y(K_Y)$. Then we can write $\deg(\omega_Y \otimes \mathcal{O}_A) = \deg(\mathcal{O}(K_Y) \otimes \mathcal{O}_A) = K_Y.A$ for all $A \in W_Y^{\geq 0}$. But note that we can choose such a K_Y with support in E only if X is Gorenstein.

(3.3.11) Proposition : Let Y be Gorenstein improvement. Then there exists a *unique* $M_Y \in W_Y \otimes \mathbb{Q}$ such that for all $A \in W_Y^{\geq 0}$ the following holds:

$$\deg(\omega_Y \otimes \mathcal{O}_A) = M_Y.A$$

One then has:

$$M_Y \underset{\sim}{w} n_*(K_{\tilde{Y}} + \tilde{\Lambda}) \underset{\sim}{w} K_Y$$

proof : This is clear. ∎

We defined M_Y only for Y Gorenstein. For *general* Y we define M_Y by

(3.3.12)

$$M_Y \underset{\sim}{w} n_*(K_{\tilde{Y}} + \tilde{\Lambda})$$

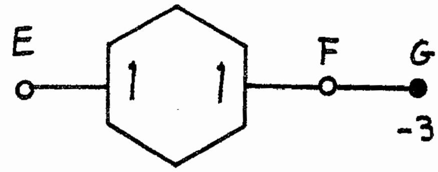
but it is not clear what it means.

What is the relation between L_Y and M_Y (for Y Gorenstein)? We have for all $s \in S$ $\alpha(s) = 2$, so the divisor P of (3.3.9) is $\tilde{\Lambda}$.
So by (3.3.9) and (3.3.11) :

$$L_Y \underset{\sim}{c} n_*(K_{\tilde{Y}} + \tilde{\Lambda}) \underset{\sim}{w} M_Y$$

hence L_Y and M_Y are Cartier equivalent, but they are different in general.

(3.3.13) **Example :** Let Y have an improvement graph that as shown below:



Then :

$$L_Y = -(1/5)(E + F) - (2/5).G$$
$$M_Y = -(1/2)(F + G)$$

(3.3.14) **Remark :** Only in the case that Y is an improvement of an X that is Gorenstein we can be sure that the cycles L_Y and M_Y coincide. Then we can find a canonical divisor K_Y with support in E so in that case:

$$K_Y = L_Y = M_Y \in C_Y$$

(3.3.15) **Names :** L_Y is called the *weakly canonical cycle.*
M_Y is called the *canonical cycle.*

Note the following formula's :

(3.3.16)

$$L_Y.A = -2.\chi(\mathcal{O}_A) - A^2 \qquad (A \in C_Y^{\geq 0})$$
$$M_Y.A = -2.\chi(\mathcal{O}_A) - A^2 - \tilde{\Delta}.A \qquad (A \in W_Y^{\geq 0})$$

The cycle L_Y and M_Y have the following positivity property:

(3.3.17) **Lemma :** Let Y be a weakly minimal improvement. Then :

$$L_Y.C \geq 0 \qquad \text{for all } C \in C_Y^{\geq 0}$$

$$M_Y.W \geq 0 \qquad \text{for all } W \in W_Y^{\geq 0}$$

proof : We only have to check to first statement for *indecomposable* Cartier divisors. There are two types of them (3.2.14);

1) The fundamental Cartier divisors F(s) at the special points.
2) The irreducible curves not intersecting Δ.

For those of the second type the statement is well known and follows immediately from (3.3.16). For those of the first type we can use (3.3.16), (3.2.4) and (3.2.6) to get:

(3.3.18)

$$L_Y.F(s) = (\alpha(s)-2) \geq 0$$

For M_Y similarly. ∎

(3.3.19) Remark : The possibilities for zero intersection are:

A. $L_Y.C = 0 \iff$ C is a (-2)-curve not intersecting Δ or C is fundamental Cartier at A_∞ or D_∞ point.

B. $M_Y.W = 0 \iff$ W is a (-2)-curve not intersecting Δ or W is a (-1)-curve intersecting Δ once (type 1).

(3.3.20) Corollary : If Y is a weakly minimal improvement, then:

$L_Y \leq 0$

$M_Y \leq 0$

proof : Write $L_Y = L_Y^+ - L_Y^-$ with L_Y^+ , $L_Y^- \in C_Y^{\geq 0} \otimes \mathbb{Q}$.

Then $L_Y.L_Y^+ = (L_Y^+)^2 - L_Y^+.L_Y^-$. If $L_Y^+ \neq 0$ then we would have $L_Y.L_Y^+ < 0$, contradicting (3.3.17).

Similarly for M_Y ∎

(3.3.21) Corollary : If Y is a weakly minimal improvement, then:

A. $\operatorname{supp}(L_Y)$ is a union of connected components of E.

B. $\operatorname{supp}(M_Y)$ is a union of connected components of $\tilde{E}$

proof : If $\operatorname{supp}(L_Y)$ would not be the union of connected components of E, then one could find an indecomposable Cartier divisor $C \not\subset \operatorname{supp}(L_Y)$, but intersecting L_Y, negatively by (3.3.20) and contradicting (3.3.17).

Similarly for M_Y ∎

Now we have to figure out how the components look like on which L_Y or M_Y have empty support.

First consider the connected components disjoint from Δ.

If L_Y or M_Y has no support on such a component, it cointains by (3.3.19) only (-2)-curves and it is well known that then this component is the minimal resolution of a *rational double point* (RDP).

Assume that we now have a component of $\tilde{Y}$ on which M_Y does not have support and which intersects $\tilde{X}$. By (3.3.19) a curve through a component of $\tilde{X}$ has to be a (-1)-curve, say A. There cannot be another (-1)-curve intersecting A, by negative definiteness of the intersection form. So only (-2)-curves can intersect A. There cannot be more than one, because otherwise we could blow down A, which brings us in the preceding situation. So there is at most one (-2)-curve intersecting A, and blowing down A makes this (-2) into a (-1)-curve so we can repeat.

conclusion: A component of $\tilde{Y}$ on which M_Y does not have support, can be blown down to a smooth germ (or to an RDP).

Similarly for L_Y: A component of Y on which L_Y does not have support, can be blown down to A_∞ or D_∞ (or to an RDP).

We summarize the above discussion into:

(3.3.22) Theorem : Let X be an AWN surface germ and let $Y \xrightarrow{\pi} X$ a Cartier improvement which is *weakly minimal.*

Let Z be a connected component of Y and F its exceptional set.

Then : $L_Z = L_Y|_Z \le 0$

$\mathrm{supp}(L_Z) = F$, except when $L_Z = 0$, in which case Z can be blown down to an RDP , A_∞ or D_∞ .

Let $\tilde{Z}$ be a connected component of $\tilde{Y}$ and $\tilde{F}$ its exceptional set.

Then : $M_{\tilde{Z}} = \tilde{M}_Y|_{\tilde{Z}} \le 0$

$\mathrm{supp}(M_{\tilde{Z}}) = \tilde{F}$, except when $M_{\tilde{Z}} = 0$, in which case $\tilde{Z}$ can be blown down to an RDP or a smooth germ.

(3.3.23) Remark : This theorem will be used in § 4.3.

The statement for isolated singularities is originally due to M. Reid. (We followed the arguments in [Do].)

§ 3.4 The Fundamental Cycle

Consider a resolution $Y \longrightarrow X$ of a *normal* surface singularity. Put $\mathcal{Z} = \{ Z \in W_Y^{>0} \mid Z.W \le 0,\ \forall\, W \in W_Y^{\ge 0} \}$. This set is non-empty by negative definiteness of the intersection form and is easily seen to be closed under "inf", so it has a unique minimal element Z, called *the fundamental cycle* of the resolution. It was introduced by Artin [Art] to study rational singularities, but was implicitly used by Du Val in [DuV]. This cycle has a number of remarkable properties. We mention a few:

1) $h^0(\mathcal{O}_Z) = \dim H^0(\mathcal{O}_Z) = 1$ (Connectedness)

2) If $\rho : Y_1 \longrightarrow Y_0$ is a modification, Z_i the fundamental cycle of Y_i, $i=0,1$, then: $\rho_* Z_1 = Z_0$, $\rho^* Z_0 = Z_1$ (Stability)

3) $\mathfrak{m}_X \cdot \mathcal{O}_Y \le \mathcal{O}_Y(-Z)$ and $\mathrm{Mult}(X,p) \ge -Z^2$ (Maximal ideal)

On an improvement of a weakly normal surface one has to be careful with the distinction between Weil and Cartier divisors. There are several 'natural' possible definitions of the fundamental cycle, which turn out to be different in general and do not have the above properties and so lose the right to be called 'fundamental'. However, after one has introduced sufficiently long *elementary chains* (see (3.2.20)), all reasonable definitions agree and give a cycle with good properties.

(3.4.1) Roots. We fix a Cartier model Y (see(3.2.14).

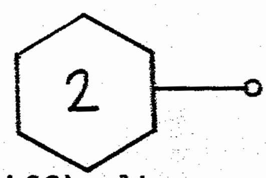
Definition : A cycle $R \in C_Y^{>0}$ is called a *root* if and only if:

$$h^0(\mathcal{O}_R) := \dim H^0(\mathcal{O}_R) = 1$$

A root R is called *rational* if $\chi(\mathcal{O}_R) = 1$, or, $h^1(\mathcal{O}_R) = 0$
elliptic if $\chi(\mathcal{O}_R) = 0$, or, $h^1(\mathcal{O}_R) = 1$, etc.

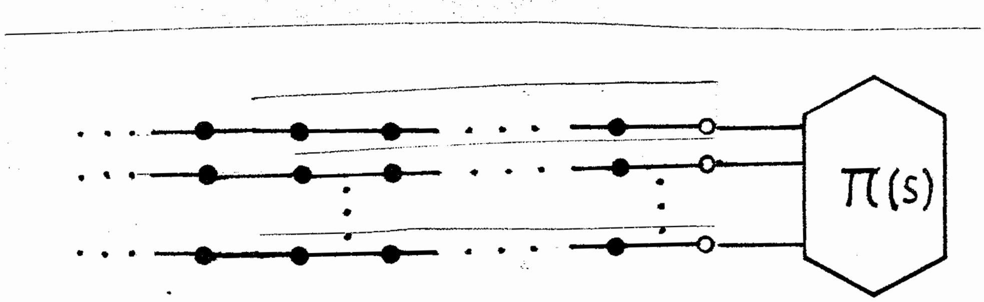
(3.4.2) **Remark :** The notion of root on a resolution appears in [Wah 2]. On the minimal resolution of an RDP these cycles correspond exactly to the positive roots of the associated root system. In any case, there are only *finitely many* roots, due to the negative definiteness of the intersection form on C_Y ($h^0(\mathcal{O}_R) = 1 \Rightarrow \chi(\mathcal{O}_R) \le 1 \Rightarrow R.(R + L_Y) \ge -2$).

In general, not every Weil divisor is contained in a root; it may even happen that the set of roots is empty, for example for:



In order to overcome this difficulty, we perform on our Cartier model Y the transformation $\eta(s)$ at all the special points $s \in S$. Around such a special point the improvement graph then contains the following curves:

(3.4.3)



We label the curves according to the following scheme:

$$\begin{array}{cccccc}
F_0^{(1)} & F_1^{(1)} & F_2^{(1)} & \cdots\cdots & F_{\alpha(1)-1}^{(1)} & F_{\alpha(1)}^{(1)} \\
F_0^{(2)} & F_1^{(2)} & F_2^{(2)} & \cdots\cdots & F_{\alpha(2)-1}^{(2)} & F_{\alpha(2)}^{(2)} \\
\vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots \\
F_0^{(k)} & F_1^{(k)} & F_2^{(k)} & \cdots\cdots & F_{\alpha(k)-1}^{(k)} & F_{\alpha(k)}^{(k)}
\end{array}$$

(3.4.4) We now introduce for every special point $s \in S$ a certain Cartier divisor $R(s)$:

$$R(s) := \Sigma_{i=1}^{k(s)} \Sigma_{j=1}^{\alpha(i,s)} j.F_j^{(i)}(s) \quad \in C_Y^{>0}$$

Note the following relations:

$$R.F_0^{(i)} = 1$$
$$R.F_j^{(i)} = 0 \quad (j = 1,2,\ldots,\alpha(i) - 1)$$
$$R.F_{\alpha(i)}^{(i)} = -1$$
$$R.R = -\alpha$$

(we suppressed the dependence on s)

(3.4.5) Lemma : For $s \in S$, $R(s)$ is a rational root. The set of indecomposable roots in $C_Y^{>0}$ consists of:

1) The *special roots* $R(s)$, $s \in S$
2) The irredicible curves E_i , not intersecting Δ.

Proof : By (3.3.18) $F(s).L_Y = \alpha(s) - 2$, where $F(s)$ is the 'fundamental' Cartier divisor at the point s. By the definition of $R(s)$ we have $R(s) = F(s) + (-2)$-curves. As $L_Y.\ (-2)\text{-curve} = 0$: $R(s).L_Y = \alpha(s) - 2$. Using (3.3.16) we find $\chi(\mathcal{O}_{R(s)}) = 1$. A neighbourhood of $R(s)$ can be blown down to a partition singularity, which is weakly rational ((2.3.7),(2.5.1)) from which it follows that $h^1(\mathcal{O}_{R(s)}) = 0$ (c.f.(4.1.3)). Hence, $R(s)$ is a rational root. It can be checked to be indecomposable. The curves under 2) are clearly indecomposable roots. ■

(3.4.6) Definition : A **root model** Z is an improvement obtained from a Cartier model Y by performing at all special points $s \in S$ of Y the transformation $\eta(s)$ of (3.2.17)

We call $\eta := \prod \eta(s) : Z \longrightarrow Y$ the root modification. (Starting with a Cartier model is not really necessary, but convenient.)

Root models have, by (3.4.5), the convenient property that their exceptional set E is covered by the supports of the indecomposable roots.

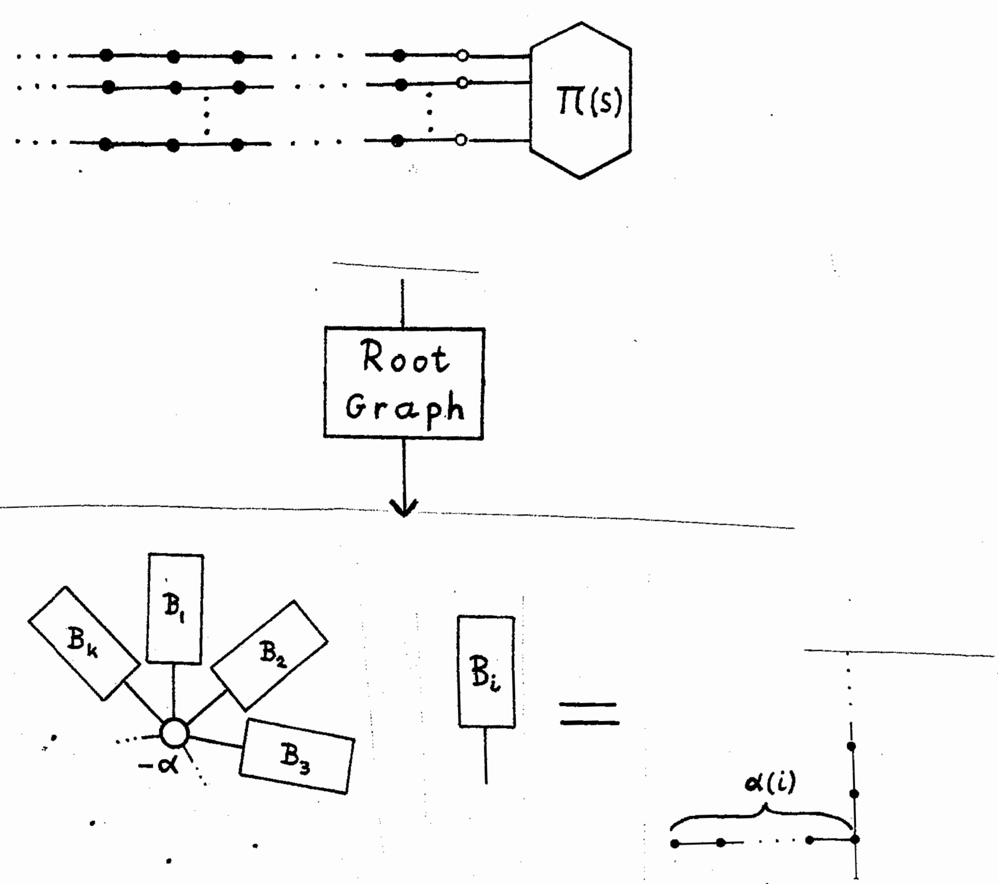
(3.4.7) **Definition :** The **root lattice** is the lattice R_Y spanned by the indecomposable roots. So we have:

$$R_Y := \oplus_{s\in S}\mathbb{Z}.R(s) \oplus \oplus\, \mathbb{Z}.E_i \ , \ E_i \cap \Delta = \emptyset$$

Remark that as *groups* we have: $R_Y = C_Y$, but their *positive cones* $R^{\geq 0}_Y$ and $C^{\geq 0}_Y$ are different in general. One has $R^{\geq 0}_Y \subseteq C^{\geq 0}_Y$.

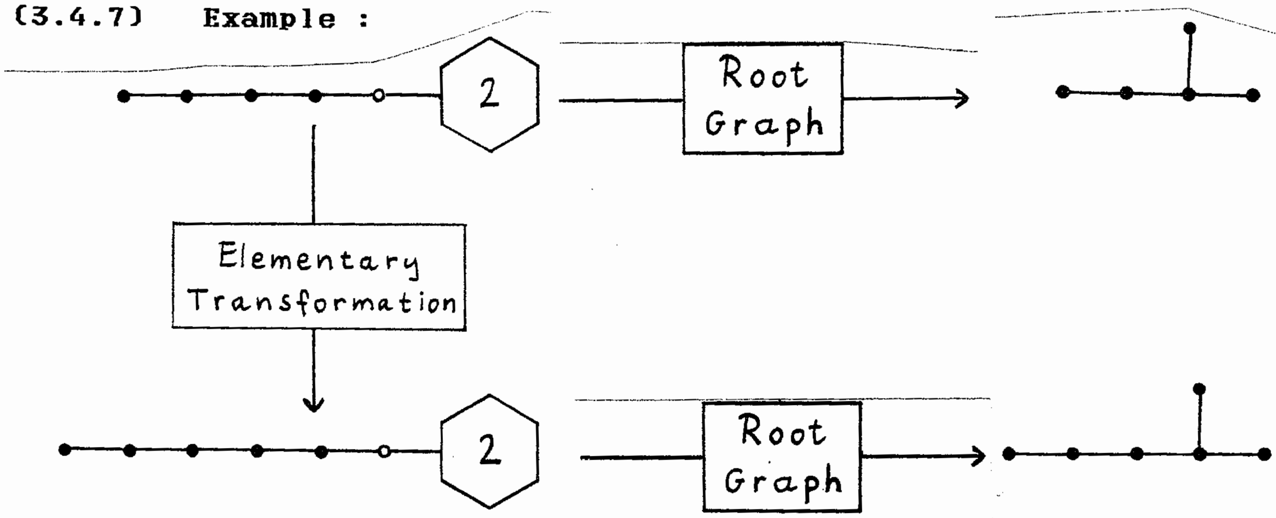
The **root graph** is the graph obtained via usual rules by taking as set of vertices the indecomposable roots. A special root R(s) counts as a smooth $\mathbb{P}^1$ with self intersection $-\alpha(s)$. On a root model as we have defined above we have that the root graph is *connected* precisely when Y is.

It is very enlightening to look at the root graph of a root model Y. Starting from the improvement graph of Y its associated root graph is obtained by performing the following pictural transformation:



Performing an elementary transformation has, on the level of the associated root graph the effect of the addition of a (-2)-curve:

(3.4.7) Example :



These associated root graphs can be considered as the *resolution graphs* of isolated singularities. The elementary transformations produce in this way a *series* of (resolution graphs of) isolated singularities. We conjecture that these graphs are resolution graphs of singularities into which the original singularity X *deforms*. (see (3.4.33) for a more precise statement). Theorem (1.3.10) can be considered as an explicit verification of this conjecture for partition singularities.
In any case, the root graph *explains* why 'formal arguments' about resolutions of normal surface singularities often carry over *verbatim* to improvements of WNCM surfaces, as soon as they can be expressed in terms of roots. This is our main motivation for the introduction of roots in this context.

(3.4.8) For an improvement Y we introduce the following sets:

$$\mathcal{Z}_W = \{ A \in C_Y^{\geq 0} \mid A.W_Y^{\geq 0} \leq 0 \}$$

$$\mathcal{Z}_C = \{ A \in C_Y^{\geq 0} \mid A.C_Y^{\geq 0} \leq 0 \}$$

$$\mathcal{Z}_R = \{ A \in C^{\geq 0} \mid A.R_Y^{\geq 0} \leq 0 \}$$

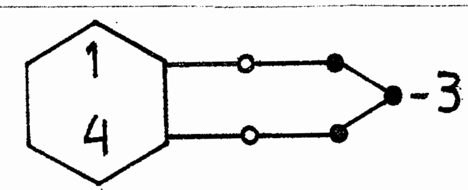
To avoid some trouble with supports of cycles, we assume from now on that Y is *connected* . (c.f. (3.3.22).)
Clearly one has $\mathcal{Z}_W \subseteq \mathcal{Z}_C \subseteq \mathcal{Z}_R$, and all elements of $\mathcal{Z}_R$ have full support. Further, these sets are closed under 'inf' , and we

define : $Z_* := \inf \mathcal{Z}_*$, $* = W, C, R$. ($\inf \emptyset := \infty$). Hence:

$$Z_W \geq Z_C \geq Z_R$$

The set $\mathcal{Z}_W$ can very well be empty, but by negative definiteness of the intersection form on C_Y the set $\mathcal{Z}_C$ is non-empty.

(3.4.9) **Example :** A. Take the following improvement graph:



$\mathcal{Z}_W = \emptyset$

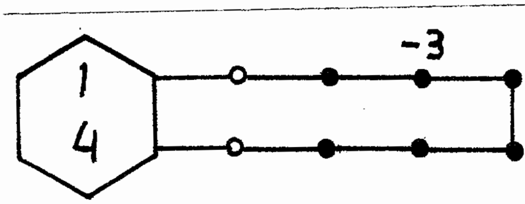
Coefficients of Z_R:

1 2

3

4 3

A root model is :

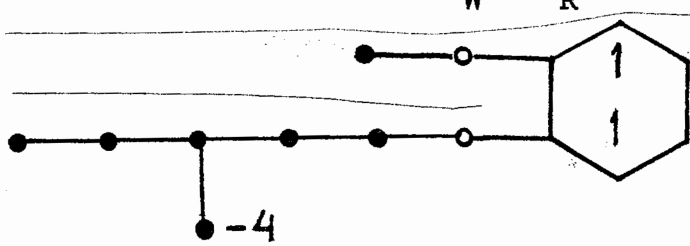


$Z_W = Z_R$

1 1 1 1

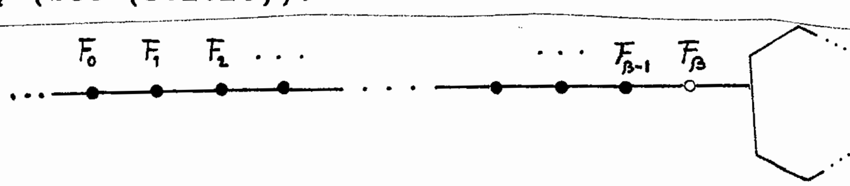
4 4 3 2

B. But even on a root model one can have $Z_W \neq Z_R$ (see (3.4.19)).



(3.4.10) **Question :** Is it true that on a root model always $\mathcal{Z}_W \neq \emptyset$? This would make life easier to deal with.

We now analyse the conditions imposed on a Cartier cycle A for negative intersection with the curves of an elementary chain. So consider such a chain of length β and label the curves in the usual way (see (3.2.20)).



Write $A = a_0.F_0 + a_1.F_1 + \ldots + a_\beta.F_\beta$ + terms not involving F_i 's.
For A to be in the set $\mathcal{Z}_R$, it is necessary that

$$A.F_i \leq 0 \quad \text{for } i = 0,1,\ldots,\beta-1$$

This is equivalent to saying that the function

$$a : \{0,1,\ldots,\beta\} \longrightarrow \mathbb{N} \; ; \; i \longmapsto a_i$$

is *concave* , i.e : $2.a_i \geq a_{i-1} + a_{i+1}$, $i = 1,2,\ldots,\beta-1$.

(3.4.11) **Lemma :** Consider the following set of functions:

$$C(a,b) := \{f:\{0,1,\ldots,a\} \longrightarrow \mathbb{N} \mid f \text{ is concave and } f(a) = b\}$$

(This set is closed under 'inf')
Put : $F_{a,b} = \inf C(a,b)$
Then :

$$F_{a,b}(n) = \begin{cases} (m+1).n & n \geq r \\ m.n + r & n \leq r \end{cases}$$

where $b = m.a + r$; $m \in \mathbb{N}$, $0 \leq r < a$.

proof : This is a nice exercise with a 'computation sequence - heat equation' argument. ∎

Note in particular that if $b = \lambda.a$, then $F_{a,b}(n) = \lambda.n$.

(3.4.12) **Lemma :** Let Y be a root model. Then :

$$\mathcal{Z}_R \subset R_Y^{>0}.$$

proof : Let $A \in \mathcal{Z}_R$ and concentrate on a special point $s \in S$.
Write A as:

$$A = \Sigma_{i=1}^{k} \Sigma_{j=1}^{\alpha(i)} a_j^{(i)} . F_j^{(i)} \quad + \text{terms not involving } F_j^{(i)}$$

(see (3.4.3)). As A is Cartier at s : $a_j^{(i)} = \lambda.\alpha(i)$ for some $\lambda \geq 0$
As $A \in \mathcal{Z}_R$, the functions:

$$a^{(i)}: \{0,1,\ldots,\alpha(i)\} \longrightarrow \mathbb{N} \quad ; \quad j \longmapsto a_j^{(i)}$$

are all concave. From lemma (3.4.11) it follows that $a_j^{(i)} \geq \lambda.j$.
Hence we can write A as a positive linear combination of roots,
i.e. $A \in R_Y^{>}$ ∎

(3.4.13) Corollary : If Y is a root model, then :

$$Z_W\ ,\ Z_C\ ,\ Z_R\ \in R_Y^{>}$$

In particular, Z_R is the fundamental cycle of the associated root graph.

(3.4.14) Computation sequences : This notion was introduced by Laufer in [Lau 2]. It is not only a convenient way to compute the fundamental cycle, but also useful for theoretical arguments.

Consider a chain of cycles $Z_0\ ,\ Z_1\ ,\ldots,\ Z_k\ ,\ldots$ of the following type:

$$Z_0 = R_0 \qquad\qquad\qquad\qquad R_0 \in \mathcal{R}$$

$$Z_k = Z_{k-1} + \mathcal{R}(W_k) \quad \text{with} \quad Z_{k-1}.W_k > 0 \qquad W_k \in \mathcal{W}$$

Here $\mathcal{W}$ denotes the set of irreducible Weil divisors, $\mathcal{R}$ the set of indecomposable roots and $\mathcal{R}(W)$ the smallest root having a support containing $W \in \mathcal{W}$.

We say that the chain *stops* at Z_k , if it cannot be extended over Z_k . Clearly if it stops at Z_k, then $Z_k \in \mathcal{Z}_W$.

Let us show that in that case $Z_k = Z_W$. This is a well known argument, but let us make sure it works in this more general situation.

We are going to show by induction that $Z_j \le Z_W$ for all terms in the chain. For j=0 this is clear, because Z_W has full support, and is by (3.4.13) a positive sum of roots. So assume $Z_{j-1} \le Z_W$.

Write:

$$Z_{j-1} = S + \alpha.\mathcal{R}(W_j) \quad , \quad S \in R_Y^{\ge 0} \text{ not involving } \mathcal{R}(W_j)$$

$$Z_W \ = T + \beta.\mathcal{R}(W_j) \quad , \quad T \in R_Y^{\ge 0} \text{ not involving } \mathcal{R}(W_j)$$

By assumption $S \le T$, $\alpha \le \beta$. Now $(Z_W - Z_{j-1}).W_j < 0$ so we get: $(T - S).W_j + (\beta - \alpha).\mathcal{R}(W_j).W_j < 0$. As $(T - S).W_j \ge 0$ and $\mathcal{R}(W_j).W_j < 0$ (this is the crucial thing) one has $\alpha < \beta$ so $Z_j = Z_{j-1} + \mathcal{R}(W_j) \le Z_W$ and we are done. Hence :

(3.4.15) Lemma : Let Y be a root model. The above computation sequence (and then stops at Z_W) iff $\mathcal{Z}_W \ne \emptyset$.

(3.4.16) Remark : There are similar computation sequences for the cycles for Z_C and Z_R . If one is not on a root model one can use the Cartier hull (3.2.15) instead of the root hull $\mathcal{R}(W)$.

An application of computation sequences is the following (c.f.[Art])

(3.4.17) Proposition : Let Y be a root model and Z_R be the fundamental cycle with respect to roots

Then : $h^0(\mathcal{O}_{Z_R}) = 1$ (i.e. Z_R is a root)

proof : Let Z_0 , Z_1 , ..., $Z_k = Z_R$ be a computation sequence for Z_R. Look at the short exact sequence:

$$0 \longrightarrow \mathcal{O}_{Z_i}(-Z_{i-1}) \longrightarrow \mathcal{O}_{Z_i} \longrightarrow \mathcal{O}_{Z_{i-1}} \longrightarrow 0$$

Now $\mathcal{O}_{Z_i}(-Z_{i-1}) \approx \mathcal{O}_R(-R.Z_{i-1})$, where $R = Z_i - Z_{i-1}$.

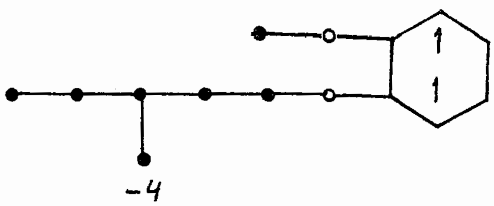
As Z_0 , Z_1 , ..., is a computation sequence : $R.Z_{i-1} > 0$. A *negative bundle over a root* cannot have sections, so we get:

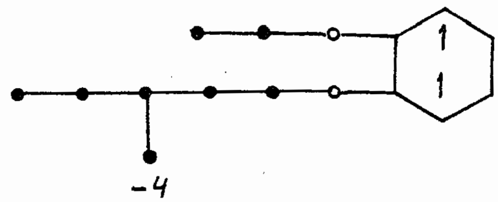
$$H^0(\mathcal{O}_{Z_i}) \longrightarrow H^0(\mathcal{O}_{Z_{i-1}})$$

and by induction the statement follows. ∎

(3.4.18) Stability : Let $\varepsilon : Y_1 \longrightarrow Y_0$ be an elementary transformation at a special point $s \in S$ of a root model Y_0 . We can compute the cycles $Z_1 = Z_R(Y_1)$ and $Z_0 = Z_R(Y_0)$. We have the induced maps ε_* and ε^* of (3.2.13) and we can ask for the relation between $\varepsilon^* Z_0$ and Z_1 or $\varepsilon_* Z_1$ and Z_0 . For a resolution it is known that $\varepsilon^* Z_0 = Z_1$; $\varepsilon_* Z_1 = Z_0$, i.e. the fundamental cycle is "stable" under blowing up and down (see [Wag]). For our cycles Z_W , Z_C , Z_R this is unfortunately not the case in general. (Remember, the elementary transformations correspond to the introduction of a (-2)-curve in the associated graph, and not to a blow up).

(3.4.19) **Example :** Take the improvement graph of example (3.4.9) B. and blow up in the short arm:





The coefficients of Z_W and Z_R look like:

$$\underset{Z_{W,0}}{\begin{array}{cccccc} & & & & 2 & 3\\ 1&2&3&3&3&3\\ & &1& & & \end{array}} \qquad \underset{\varepsilon^* Z_{W,0}}{\begin{array}{cccccc} & & & 2 & 3 & 3\\ 1&2&3&3&3&3\\ & &1& & & \end{array}} \quad > \quad \underset{Z_{W,1}}{\begin{array}{cccccc} & & & 1 & 2 & 3\\ 1&2&3&3&3&3\\ & &1& & & \end{array}}$$

$$\underset{Z_{R,0}}{\begin{array}{cccccc} & & & & 1 & 2\\ 1&2&3&3&3&2\\ & &1& & & \end{array}} \qquad \underset{\varepsilon^* Z_{R,0}}{\begin{array}{cccccc} & & & 1 & 2 & 2\\ 1&2&3&3&3&2\\ & &1& & & \end{array}} \quad < \quad \underset{Z_{R,1}}{\begin{array}{cccccc} & & & 1 & 2 & 3\\ 1&2&3&3&3&3\\ & &1& & & \end{array}}$$

So neither Z_W nor Z_R is stable and they exhibit opposite behaviour. After the blow up they have become equal!(and stable).

To analyse stability it is most easy to use Z_W . So take a model Y_0 on which Z_W exists (see (3.4.27)) and let $Y_1 \xrightarrow{\varepsilon} Y_0$ be an elementary modification. Put $Z_i = Z_W(Y_i)$ $i = 0 , 1$. Let D the newly introduced curve, and let ε^*A be the total, $\bar{A}$ the strict transform of $A \in W_{Y_0}^{\geq 0}$.

We use the formulas $\varepsilon^*A = \bar{A} + (\bar{A}.D).D$; $\varepsilon^*A.D = 0$.

As $\varepsilon^*\colon C_{Y_0} \longrightarrow C_{Y_1}$, we have $\varepsilon^*Z_0 \in C_{Y_1}^{>0}$. We don't necessarily have $\varepsilon_*Z_1 \in C_{Y_0}$.

Now: $$\left.\begin{array}{l} \varepsilon^*Z_0.\bar{W}_i = \varepsilon^*Z_0.\varepsilon^*W_i = Z_0.W_i \\ \varepsilon^*Z_0.D = 0 \end{array}\right\} \Rightarrow \quad \varepsilon^*Z_0 \geq Z_1$$

Write: $\varepsilon^{*}Z_0 = Z_1 + P,\ P \in C^{\geq}_{Y_1}$.

Then: $Z_0^2 = (\varepsilon^{*}Z_0)^2 = Z_1^2 + 2.Z_1.P + P^2 \leq Z_1^2$ as $Z_1.P$, $P^2 \leq 0$.

So: $|Z_0^2| \geq |Z_1^2|$ with equality if and only if $\varepsilon^{*}Z_0 = Z_1$.

Write: $Z_1 = \bar{\bar{Z}} + \alpha.D \qquad Z = \varepsilon_{*}Z_1$

Then:

$$\left.\begin{array}{l} \varepsilon^{*}\varepsilon_{*}Z_1 = \bar{\bar{Z}} + (\bar{\bar{Z}}.D).D \\ \left.\begin{array}{l} Z_1 = \bar{\bar{Z}} + \alpha.D \\ Z_1.D \end{array}\right\} \Rightarrow \alpha \geq \bar{\bar{Z}}.D \end{array}\right\} \Rightarrow Z_1 = \varepsilon^{*}\varepsilon_{*}Z_1 + \underbrace{(\alpha - \bar{\bar{Z}}.D)}_{\geq 0}.D$$

Now: $\varepsilon_{*}Z_1.W_i = \varepsilon^{*}\varepsilon_{*}Z_1.\varepsilon^{*}W_i = Z_1.\varepsilon^{*}W_i \leq 0$.

So: *If* $\varepsilon_{*}Z_1$ is Cartier, then $\varepsilon_{*}Z_1 \geq Z_0$. If this is the case then

$\varepsilon^{*}\varepsilon_{*}Z_1 \geq \varepsilon^{*}Z_0 \geq Z_1 \geq \varepsilon^{*}\varepsilon_{*}Z_1$

(3.4.20) Resume :

1) $\varepsilon^{*}Z_0 \geq Z_1$; $|Z_0^2| \geq |Z_1^2|$, = iff $\varepsilon^{*}Z_0 = Z_1$

2) If $\varepsilon_{*}Z_1 \geq Z_0$, then $\varepsilon^{*}Z_0 = Z_1$ and $\varepsilon_{*}Z_1 = Z_0$

3) $\varepsilon_{*}Z_1.W^{\geq 0}_{Y_0}$, so if $\varepsilon_{*}Z_1$ is Cartier, then $\varepsilon_{*}Z_1 \geq Z_0$

(3.4.21) Corollary : (If Z_W exists on Y then) after finitely many elementary transformations Z_W becomes stable.

We now give a criterion for the stability of Z_W in terms of the lengths of the elementary chains (see (3.2.20). Assume Y_0 is a root model on which $Z_W = Z_0$ exists. Let $s \in S$ be a special point of Y_0 and let $\lambda(s)$ be the *coefficient* of $R(s)$ in Z_0.
Let $Y_1 \xrightarrow{\rho} Y_0$ a model dominating Y_0 and let $Z_1 = Z_W(Y_1)$

(3.4.22) Lemma : Let s' be the special point of Y_1 corresponding to s on Y_0. If all elementary chains terminating at s' have length $\geq \lambda(s)$.Index, then Z_1 is stable under further elementary transformations at s.

proof : By (3.4.20) we have $\rho^* Z_0 \geq Z_1$ so $\lambda(s) \geq \lambda'(s')$, where $\lambda'(s')$ is the coefficient of the special root at s' in Z_1. Let $\{F_0, F_1, F_2, \ldots, F_\beta\}$ be an elementary chain on Y_1 terminating at s'. By assumption : $\beta \geq \lambda(s).\alpha(i,s') \geq \lambda'(s').\alpha(i,s')$. Writing the cycle Z_1 as:

$$Z_1 = a_0.F_0 + a_1.F_1 + \ldots + a_{\beta-1}.F_{\beta-1} + a_\beta.F_\beta + \text{other terms}$$

we see that by (3.4.11) that if $\beta > a_\beta = \lambda'(s').\alpha(i,s')$ then $a_{\beta-1} = a_\beta$ and by (3.4.20) 3) it then follows that Z_1 is stable under further elementary transformations at s'. ■

(3.4.23) Remark : An Y_1 as above can be obtained from Y_0 by performing successively $\lambda(s)$ times the operation $\eta(s)$ at all special points.

(3.4.24) Lemma : Let $Y_1 \xrightarrow{\rho} Y_0$ be as in (3.4.22). Then $Z_R(Y_1) = Z_W(Y_1)$.

proof : We always have $Z_R(Y_1) \leq Z_W(Y_1)$. From this it follows that the coefficient of the special roots in $Z_R \leq \lambda(s)$. Using (3.4.11) again, one can check that $Z_R.W \leq 0$ for all $W \in W^{\geq 0}_{Y_1}$, i.e Z_R fulfils the conditions for Z_W so $Z_R \geq Z_W$ ■

(3.4.25) Fundamental cycle and Maximal ideal.

One should not forget that the fundamental cycle is intended to be a *lattice theoretical* and hence computable substitute for the maximal ideal $\mathfrak{m}_X$ of a surface singularity. Let $Y \xrightarrow{\pi} X$ be an improvement of an AWN-surface germ X and let $\mathfrak{m}_X = \mathfrak{m}_{X,p}$ be the maximal ideal of the local ring $\mathcal{O}_{X,p}$. We can pull back the functions in $\mathfrak{m}_X$ to Y, where they generate a sheaf which we denote by $\mathfrak{m}_X.\mathcal{O}_Y \subset \mathcal{O}_Y$. Take a function $f \in \mathfrak{m}_X$ and look at the divisor of $\pi^* f \in H^0(\mathfrak{m}_X.\mathcal{O}_Y)$. In general it will consist of a non-compact part N and a part C with support on E : $(\pi^* f) = N + C$. Now assume that $N \cap \Delta = \emptyset$. Then C is a Cartier divisor. Clearly we have $0 = (\pi f).W_i = C.W_i + N.W_i$, hence $C.W_i \leq 0$ for all irreducible Weil divisors on Y and so $C \geq Z_W$ (and in particular the set $\mathcal{Z}_W$ is non-empty). If $\mathfrak{m}_X.\mathcal{O}_Y$ is an *invertible* ideal at the special points $s \in S$ we certainly can find such an f, so :

(3.4.26) **Lemma :** If $\mathfrak{m}_X \cdot \mathcal{O}_Y$ is invertible at S, then:

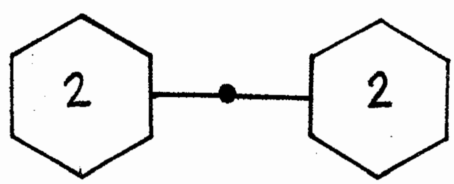
1) $\mathcal{Z}_W \neq \emptyset$, i.e. Z_W exists.
2) $\mathfrak{m}_X \cdot \mathcal{O}_Y \subseteq \mathcal{O}_Y(-Z_W)$

As to be expected this need not be the case (apparently (3.4.9) A. is an example) To see this in an explicit example, we take the following

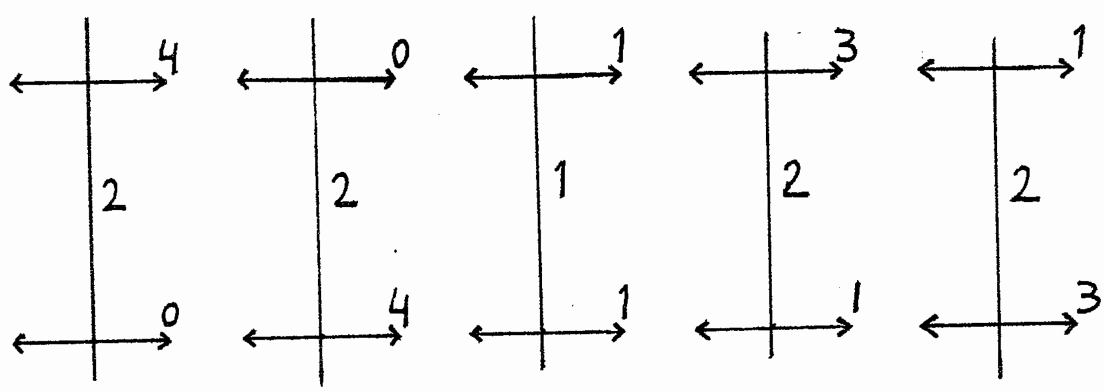
(3.4.27) **Example :** Take the singularity X with

$$\mathcal{O}_{X,p} = \mathbb{C}\{x^4,\ y^4,\ x.y,\ x^3.y,\ x.y^3\}.$$

This is a weakly rational quadruple point in $\mathbb{C}^5$. Its normalization is an A_1 point and its (minimal) improvement has the following graph:



The divisors of the functions in $\mathfrak{m}_X$ on $\tilde{Y}$ look like:



x^4 y^4 $x.y$ $x^3.y$ $x.y^3$

In local coordinates around the point $\tilde{s}$ we have :

$\mathfrak{m}_X \cdot \mathcal{O}_{\tilde{Y},\tilde{s}} = (u.v,\ u^2,\ u^2v^4,\ u^2v,\ u^2v^3)$, hence $\mathfrak{m}_X \cdot \mathcal{O}_Y$ is not

invertible at s, and indeed $\mathfrak{M}_X.\mathcal{O}_Y \not\subset \mathcal{O}_Y(-Z_W) = \mathcal{O}_Y(-2.E)$
Blowing up once in the special points $\tilde{s}$ gives a model on which :

$$\mathfrak{M}_X.\mathcal{O}_Y = \mathcal{O}_Y(-Z_W) = \mathcal{O}_Y(- E - 2.F - 2.G)$$

where F, G are the newly introduced curves.

(3.4.28) Definition : An improvement $Y \xrightarrow{\pi} X$ is called a *stable model* if and only if the following conditions are fulfiled:

1) Y is a root model
2) $Z_R = Z_W$
3) Z_R is stable

For such a stable model we simply write $Z = Z_R = Z_W$ and call this cycle the fundamental cycle.

(3.4.29) Theorem : Let X be an AWN surface germ. Then there exists a stable model $Y \xrightarrow{\pi} X$ for X.
Let Z be its fundamental cycle. Then:

1) $h^0(\mathcal{O}_Z) = 1$
2) Z is stable under all modifications.
3) $\mathfrak{M}_X.\mathcal{O}_Y \leq \mathcal{O}_Y(-Z)$; $\mathrm{Mult}(X,p) \geq -Z^2$

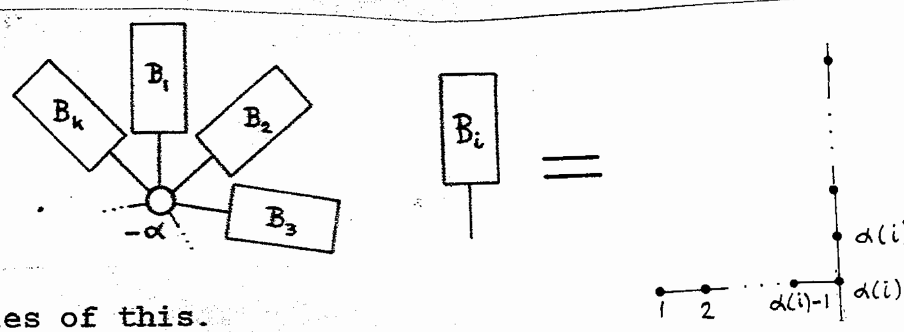
proof: First take a root model (3.4.6). Then make $\mathfrak{M}_X.\mathcal{O}_Y$ invertible at the special points (3.4.26) so that Z_W exists. Then perform, if necessary, further elementary transformations to make Z_W stable and equal to Z_R (3.4.22) , (3.4.24). The resulting model will then be a stable model.
Now let an arbitrary stable model $Y \longrightarrow X$ be given. It is clear that its fundamental cycle Z fulfils 1) (by 3.4.17) and 2) (almost by definition). For 3) let $Y' \xrightarrow{\rho} Y$ be a model dominating Y with $\mathfrak{M}_X.\mathcal{O}_{Y'}$ invertible around the special points and Z' its fundamental divisor. So $\mathfrak{M}_X.\mathcal{O}_{Y'} \leq \mathcal{O}_{Y'}(-Z')$. By stability we have $Z' = \rho^*Z$, $\rho_*Z' = Z$, so:

$$\rho^*\mathcal{O}_Y(-Z) = \mathcal{O}_{Y'}(-Z') \quad ; \quad \rho_*\mathcal{O}_{Y'}(-Z') = \mathcal{O}_Y(-Z)$$

Hence $\mathfrak{M}_X.\mathcal{O}_Y \longrightarrow \rho_*\rho^*\mathfrak{M}_X.\mathcal{O}_Y \leq \rho_*\mathfrak{M}_X.\mathcal{O}_{Y'} \leq \rho_*\mathcal{O}_{Y'}(-Z') = \mathcal{O}_Y(-Z)$ ∎

(3.4.30) **Remark :** We see that in any case by performing finitely many elementary transformations on a given improvement we end up with a stable model on which the fundamental cycle has formally the same properties as in the case of a resolution of a normal surface singularity. But the exact number of blow-ups needed is still very mysterious. If (3.4.10) has an affirmative answer, we would have an explicit upper bound for this number. This is now lacking, because we used an unknown number of modifications to make $\mathfrak{m}_X.\mathcal{O}_Y$ invertible at S to get the existence of Z_W . Although this is not very satisfactory, we will not pursue this further, because for most applications the above results suffice.

(3.4.31) **Remark :** Sometimes it is convenient to blow up a stable model still further to what I would like to call a *very stable model.* On such a very stable model the coefficients of Z on the curves near the special points are *constant* on elementary chains of length equal to their index. So on the associated root graph the coefficients of Z are:



or multiples of this.

(3.4.32) **Associated root graph and deformations.**

When we perform elementary transformations on a A_∞ or D_∞ singularity, and then look at the corresponding root graph we get the picture as in (3.4.7). Now A_∞ and D_∞ are usually considered as the "limits" of the series of isolated singularities A_n and D_n. So here the associated root graphs are precisely the resolution graphs of the series of isolated singularities into which the limit *deforms*. By theorem (1.3.12) the same is true for the other partition singularities.

The following conjecture seems plausible:

(3.4.33) **Conjecture :** Let (X,Σ,p) be a (germ of a) WNCM-surface. Then there exists an improvement $Y \longrightarrow X$ with the following property:

For every root graph Γ a of model $Y' \longrightarrow Y$ dominating Y there exists a flat deformation $F(\Gamma) : \mathcal{X} \longrightarrow S$ of X over a smooth curve germ $(S,0)$ with the following properties:

1) The fibres $X_t = F(\Gamma)^{-1}(t)$ $(t \neq 0)$ have a single isolated point which have a resolution with resolution graph Γ.
2) $p_g(X_t) = p_g(X)$.
3) $\mathrm{Mult}(X_t,p) = \mathrm{Mult}(X,p)$.

One can also conjecture the existence of deformations between the X_t corresponding to different Γ according to inclusion of graphs. So every WNCM X should give rise to a (multi-) series of singularities, indexed by associated root graphs.
At this moment of writing I cannot give a complete proof of this conjecture. But it should be possible to proceed as follows:

1) Take an arbitrary improvement $Y \longrightarrow X$, and choose a cycle A that "carries the cohomology", i.e. a cycle A such that the natural restriction map $R^1\pi_*\mathcal{O}_Y \longrightarrow H^1(\mathcal{O}_A)$ is an isomorphism.
2) It is possible to develop a deformation theory of Y, *fixing* the subspace A, i.e. one can consider those deformations of Y inducing the trivial deformation on A. The resulting functor I call Fix_A . It has a tangent space $\mathbb{F}\mathrm{ix}^1_A$ and an obstruction space $\mathbb{F}\mathrm{ix}^2_A$. As deformations of this type are 'of local nature' , there are corresponding sheaves $\mathcal{F}\mathrm{ix}^1_A$ and $\mathcal{F}\mathrm{ix}^2_A$, which have support on the set Δ.
3) By [Siu] $H^2(\mathcal{G}) = 0$ for every coherent sheaf on Y, so the 'local to global' spectral sequence reduces to the following :a. an exact sequence

$$0 \longrightarrow H^1(\Theta_Y(-A)) \longrightarrow \mathbb{F}\mathrm{ix}^1_A \longrightarrow H^0(\mathcal{F}\mathrm{ix}^1_A) \longrightarrow 0$$

where $\Theta_Y(-A)$ is the sheaf of vectorfields, vanishing on A.

b. an isomorphism

$$\mathbb{F}ix_A^2 \longrightarrow H^0(\mathcal{F}ix_A^2)$$

4) As we already mentioned, the sheaves $\mathcal{F}ix_A^1$ and $\mathcal{F}ix_A^2$ are concentrated on Δ. One can construct deformations of the neighbourhoods of the partition singularities that fix the part of A in that neighbourhood by choosing the N_i of theorem (1.3.12) sufficiently high. This gives an element of the group $H^0(\mathcal{F}ix_A^1)$. By the above exact sequence 3) a. this element can be lifted to an element of $\mathbb{F}ix_A^1$, i.e we have found an infinitesimal deformation of Y, fixing A (with specific behaviour at the special points of Y). By 3) b. this element is not obstructed, because the global obstructions come from the local ones, and we started with local deformations that where unobstructed. This way we get a *formal deformation* "$\mathcal{Y}$" $\longrightarrow$ spec($\mathbb{C}[[t]]$)) of Y.

5) Problem: Can we do this convergent, so get a *real space* $\mathcal{Y} \longrightarrow S$ over a curve germ ? There should be a general theorem which garantees this. So assume for the moment we know that we have a real deformation $\mathcal{Y} \xrightarrow{\Pi} S$. By construction , this deformation of Y is trivial on A, and as A carries the cohomology, it follows that $t \longmapsto H^1(\mathcal{O}_{Y_t})$ is *constant*, where Y_t is the fibre $\Pi^{-1}(t)$.

6) Now there is a general theorem of Riemenschneider [Ri] stating that in such a situation ($H^1(\mathcal{O}_Y)$ constant) we can blow down $\mathcal{Y} \longrightarrow S$ to a get a $\mathcal{X} \longrightarrow S$, which is a *flat deformation* of our original singularity X.

What does a resolution of the general fibre look like? Well, first it is dominated by the deformed improvement. The local deformations were of such a type that we introduced isolated singularities at the special points. Resolving these just gives the associated root graph of a model $Y' \longrightarrow X$. It is now easy to see that $Y' \longrightarrow X$ is an improvement having the properties of the improvement of the conjecture (where it was called Y). So at least formally the conjecture is true. (The multiplicity question can be handled.) I hope to settle this convergence question in the near future.

CHAPTER 4

APPLICATIONS

In this last chapter we give some applications of the ideas and techniques developed in the foregoing chapters. The basic philosophy is: "Every formal argument about resolutions of normal surface singularities should have its counterpart for (stable) improvements of WNCM-surfaces". This makes an easy game: take a theorem about normal surface singularities, read carefully its proof and usually you will find an interesting theorem about WNCM-surfaces as well. Because we mainly have to harvest the crop sown by others, we do not give full proofs.

§ 4.1 Weakly rational singularities

Rational surface singularities have been studied thoroughly by many authors during the last decades. We mention Artin [Art], Brieskorn [Br], Tjurina [Tj], Lipman [Lip], Laufer [Lau 2] and Wahl [Wah 1]. We give a short overview of some results, stating their theorems in the more general context of weakly rational surfaces.

(4.1.1) **Definition :** Let (X,Σ,p) be a WNCM-surface germ. Then X is called *weakly rational* if and only if:

$$R^1\pi_*\mathcal{O}_Y = 0$$

where $Y \xrightarrow{\pi} X$ is any improvement.
We refer to such singularities as WNR-surface germs.

(4.1.2) **Remark :** This is the same (for X CM) as (2.6.1).
We will choose for X a small Stein representative and $Y = \pi^{-1}(X)$, so that $H^1(\mathcal{O}_Y) = (R^1\pi_*\mathcal{O}_Y)_p$.

Note that for any singularity and $A \in W_Y^{\geq 0}$ one has a *surjection:*

$$H^1(\mathcal{O}_Y) \longrightarrow H^1(\mathcal{O}_A)$$

and from this:

$$H^1(\mathcal{O}_Y) = 0 \quad \Leftrightarrow \quad H^1(\mathcal{O}_A) = 0 \ \ \forall\, A \in W_Y^{\geq 0}$$

In particular, for an improvement of a weakly rational X we can remark directly the following things:

1) All irreducible components E_i of E are smooth rational curves.
2) These components intersect transversally.
3) The improvement graph is a tree.

Artin's first remarkable theorem is that the weak rationality of a singularity can be decided by looking only to the improvement graph. It states that the fundamental cycle is big enough to detect $H^1(\mathcal{O}_Y)$.

(4.1.3) Theorem ([Art]) : Let X be a WNCM-surface germ and $Y \xrightarrow{\pi} X$ be a stable improvement of X, Z its fundamental cycle.
Then X is *weakly rational* if and only if $H^1(\mathcal{O}_Z) = 0$.

(4.1.4) Remark : As $H^0(\mathcal{O}_Z) = 1$, the condition $H^1(\mathcal{O}_Z) = 0$ is the same as $\chi(\mathcal{O}_Z) = 1$; as χ's can be computed from the improvement graph, the WNR-condition is decidable. In fact one may work on a root model when one uses Z_R .
When one looks at a computation sequence for Z:

$$Z_j = Z_{j-1} + R_j \ , \ Z_{j-1}.R_j > 0 \ , \ R_j \in \mathcal{R}$$

one sees $\chi(\mathcal{O}_Z) = 1 \quad \Leftrightarrow \quad Z_{j-1}.R_j = 1$ for all steps in the computation sequence. This is very convenient for actual computations.(This remark is due to Laufer [Lau 2].)

We have seen that on a stable model one always has:

$$\mathfrak{m}_X.\mathcal{O}_Y \leq \mathcal{O}_Y(-Z)$$

Artin's second theorem is that for weakly rational singularities one has *equality*.

(4.1.5) **Theorem ([Art],[Lip]) :** Let X be a WNR-surface germ and let $Y \xrightarrow{\pi} X$ be a stable improvement and Z its fundamental cycle. Then:

$$\mathfrak{m}_X^n . \mathcal{O}_Y = \mathcal{O}_Y(-n.Z)$$

$$\mathfrak{m}_X^n / \mathfrak{m}_X^{n+1} \approx H^0(\mathcal{O}_Z(-n.Z))$$

(4.1.6) **Corollary :** For a WNR-surface germ one has:

$$\dim(\mathfrak{m}_X^n / \mathfrak{m}_X^{n+1}) = -n.Z^2 + 1.$$

From this we get:

$$\mathrm{Embdim}(X,p) = \dim(\mathfrak{m}_X / \mathfrak{m}_X^2) = - Z^2 + 1$$

$$\mathrm{Mult}(X,p) = \lim_{n \to \infty} \left((1/n).\dim(\mathfrak{m}_X^n / \mathfrak{m}_X^{n+1}) \right) = - Z^2$$

(4.1.7) **Remark :** This theorem is certainly wrong when one does not go to a stable model (see examples (1.4.16) and (3.4.27)). This is one of our motivations for the introduction of stable models.

The next theorem is due to Tjurina:

(4.1.8) **Theorem ([Tj]) :** Let X be a WNR-surface germ and let $Y \xrightarrow{\pi} X$ be a stable improvement. Let $B(X) \longrightarrow X$ be the blow up of X at the point p. Then B(X) is obtained from Y by blowing down all maximal connected graphs of curves E_i with $E_i.Z = 0$.

(4.1.9) **Corollary :** Weakly rational surfaces can be improved by successively blowing up points. Here one has to be a bit careful, because for isolated rational singularities one uses induction on the number of curves in the minimal resolution. As minimal improvements are not stable in general, one cannot use this. But instead one can use induction to the number of curves on a weakly minimal improvement that do not occur in an elementary chain. So this is the analogue of the absolute isolatedness of rational singularities. J.Stevens told me, that from this result it follows that a general WNCM-surface germ X can be improved by blowing up points and "partial normalizations" . This is an analogue to the resolution process of Zariski we mentioned in (1.4.1).

A lot can be said about the set of equations defining a weakly rational surface singularity X. We have seen that $\mathrm{Embdim}(X,p) = -Z^2 + 1 =: e$, so $X \longrightarrow \mathbb{C}^e$. Let $\mathcal{O} = \mathcal{O}_{\mathbb{C}^e,0}$, $\bar{\mathcal{O}} = \mathrm{Gr}_m(\mathcal{O})$ and $\bar{X} = \mathrm{Specan}(\mathrm{Gr}_m(\mathcal{O}_X)$ be the tangent cone of X. The following theorem is due to J.Wahl:

(4.1.10) Theorem ([WAH 1]) : $\mathcal{O}_X$ and $\mathcal{O}_{\bar{X}}$ have minimal free resolutions of the form:

$$0 \longrightarrow \mathcal{O}^{b_{e-2}} \xrightarrow[\phi_{e-2}]{} \cdots \longrightarrow \mathcal{O}^{b_2} \xrightarrow[\phi_2]{} \mathcal{O}^{b_1} \xrightarrow[\phi_1]{} \mathcal{O} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

$$0 \longrightarrow \bar{\mathcal{O}}^{b_{e-2}} \xrightarrow[\bar{\phi}_{e-2}]{} \cdots \longrightarrow \bar{\mathcal{O}}^{b_2} \xrightarrow[\bar{\phi}_2]{} \bar{\mathcal{O}}^{b_1} \xrightarrow[\bar{\phi}_1]{} \bar{\mathcal{O}} \longrightarrow \mathcal{O}_{\bar{X}} \longrightarrow 0$$

such that : 1) the second resolution is the graded complex of the first.

2) $\bar{\phi}_i$ is homogeneous of degree 1 $(i>1)$ or 2 $(i=1)$.

3) $b_i = i.\binom{e-1}{i+1}$

A proof can be found in the beautyful paper [Wah 1].
An immediate corollary is:

(4.1.11) Corollary : The Gorenstein type of X is $\mathrm{type}(X,p) = e-2 = -Z^2 - 1$.

It should be clear by now that instead of looking for similarities between isolated rational and non-isolated weakly rational singularities, one should concentrate on the differences between them, in particular on the structure of the singular locus.
But before doing so, we proof a theorem that was used in § 2.5 (2.5.10)

(4.1.12) Theorem : Let $(X,p) \longrightarrow (S,0)$ be a flat deformation of a reduced curve germ (C,p) over a smooth curve germ $(S,0)$ (In other words, C is a general hyperplane section of X through p). Then :

(C,p) is weakly normal (i.e $(C,p) \approx (L^r_r,0)$ for some r)

if and only if

(X,p) is a weakly rational WNCM-germ, which has a *reduced fundamental cycle* on a stable weakly minimal improvement.

proof : If (C,p) is weakly normal, then by (2.5.7) (X,p) is weakly rational and WNCM. Now choose a stable model $Y \xrightarrow{\pi} X$ and let Z be its fundamental cycle. Let $\bar{C}$ be the strict transform of C on Y. We may assume that $\bar{C}$ is disjoint from the special roots by performing elementary modifications.
If $f \in \mathfrak{M}_X$ defines C, then we can write $(\pi^* f) = Z + \bar{C}$. So for all indecomposable roots $R \in \mathcal{R}$ one has: $Z.R = -\bar{C}.R$. As C consists by assumption of smooth branches, we must have: $Z.R < 0 \Rightarrow Z$ reduced at R. Let T be the sum of all indecomposable roots, $T = \sum R_i$, $R_i \in \mathcal{R}$. We show that $Z=T$. Assume $Z > T$, then the computation sequence has to terminate at a root R_i with $Z.R_i = 0$. But because X is weakly rational, remark (4.1.4) gives $(Z-R_i).R_i = 1$, so $R_i^2 = -1$. By weak minimality of Y, there cannot be such roots, hence $Z = T$. It is easy to see that the cycle T can only be the fundamental cycle if the partition singularities of Y are all of type $\pi = (1,1,\dots,1)$ (local argument). Hence, the fundamental cycle, considered as a subspace of Y, is reduced. This proves half of the theorem. The converse is easy and follows for instance from:

(4.1.13) Proposition : Let (X,p) be a weakly rational WNCM-surface germ. Let (C,p) be the germ of a general hyperplane section through p . Then:

$$\delta(C,p) = \mathrm{Mult}(X,p) - 1$$

We omit the proof, which is not hard.

So if C has r branches and Y has a reduced fundamental cycle, then $\mathrm{Mult}(X,p) = -Z^2 = Z.\bar{C} = r$, so $\delta(C,p) = r - 1$, which implies that C is a weakly normal curve germ (see (1.1.6)) ∎

(4.1.14) Remark : The proof of (4.1.11) was inspired by J.Stevens (see [Stev 1], example (3.5)).
The fact that for X as in (4.1.12) all branches of Σ are smooth and that the map $\tilde{\Delta} \longrightarrow \Delta$ is unramified, has been used in (2.5.11).

(4.1.15) What can be said about the singular locus Σ of a weakly rational surface germ X ?
By (2.3.5) and (2.3.6) the classification of weakly rational WNCM-surface germs is equivalent to the classification of:

1) Finite mappings between curve germs $\tilde{\Sigma} \longrightarrow \Sigma$ such that

   a. $H^0_{\{p\}}(\mathcal{O}_{\tilde{\Sigma}}/\mathcal{O}_{\Sigma}) = 0$ and b. $\delta(\tilde{\Sigma}) = \delta(\Sigma)$

2) Embeddings $\tilde{\Sigma} \longrightarrow \tilde{X}$ where $\tilde{X}$ is a rational surface singularity.

Let us start with condition 1). The normalizations of Σ and $\tilde{\Sigma}$ are denoted by Δ and $\tilde{\Delta}$ respectively, as usual. We can consider $\mathcal{O}_{\Sigma}$, $\mathcal{O}_{\tilde{\Sigma}}$, $\mathcal{O}_{\Delta}$ all as subrings of $\mathcal{O}_{\tilde{\Delta}}$. In (1.2.22) the condition about the local cohomology was translated in:

$$\mathcal{O}_{\tilde{\Sigma}} \cap \mathcal{O}_{\Delta} = \mathcal{O}_{\Sigma}$$

The condition $\delta(\tilde{\Sigma}) = \delta(\Sigma)$ can then be translated into:

$$\mathcal{O}_{\tilde{\Sigma}} + \mathcal{O}_{\Delta} = \mathcal{O}_{\tilde{\Delta}}$$

This is because $\delta(\Sigma) = \dim(\mathcal{O}_{\Delta}/\mathcal{O}_{\Sigma}) = \dim(\mathcal{O}_{\Delta}/\mathcal{O}_{\tilde{\Sigma}}\cap\mathcal{O}_{\Delta}) = \dim((\mathcal{O}_{\tilde{\Sigma}}+\mathcal{O}_{\Delta})/\mathcal{O}_{\tilde{\Sigma}})$ and $\delta(\tilde{\Sigma}) = \dim((\mathcal{O}_{\tilde{\Sigma}}+\mathcal{O}_{\Delta})/\mathcal{O}_{\tilde{\Sigma}}) \iff \mathcal{O}_{\tilde{\Sigma}} + \mathcal{O}_{\Delta} = \mathcal{O}_{\tilde{\Delta}}$.

From these two conditions already a lot can be said about Σ and $\tilde{\Sigma}$.

(4.1.16) Proposition : Let $\tilde{\Sigma} \longrightarrow \Sigma$ be a map between curve (multi-) germs with the property that

$$\mathcal{O}_{\Delta} \cap \mathcal{O}_{\tilde{\Sigma}} = \mathcal{O}_{\Sigma} \qquad \text{and} \qquad \mathcal{O}_{\Delta} + \mathcal{O}_{\tilde{\Sigma}} = \mathcal{O}_{\tilde{\Delta}}$$

Then:

A. Any component of $\tilde{\Delta}$ maps to a smooth component of $\tilde{\Sigma}$, or maps with degree 1 to a component of Σ (or both).

B. If Σ is an irreducible singular curve, then $\tilde{\Sigma}$ consists of a disjoint union of smooth branches L_i , together with a copy of the curve Σ: $\tilde{\Sigma} = \coprod L_i \coprod \Sigma$. The map $\tilde{\Sigma} \longrightarrow \Sigma$ is the identity on the Σ component; the map $L_i \longrightarrow \Sigma$ is the normalization map composed with a cyclic covering.

C. If $\Sigma = L_r^r$, then $\tilde{\Sigma}$ is the disjoint union $\coprod L_s^s$. All branches of $\tilde{\Sigma}$ map with a certain degree to components of Σ. One can associate in a natural way an incidence graph to this situation, which has to be a tree.

D. If $\tilde{\Sigma}$ is connected and all components map with degree > 1 to components of Σ , then Σ and $\tilde{\Sigma}$ are weakly normal.

We omit the proof, which is straightforward.

(4.1.17) Corollary : An irreducible curve Σ can be the singular locus of a weakly rational surface if and only if Σ can be embedded on a rational surface singularity. This follows from case B. of (4.1.16).

(4.1.18) Corollary : If X is an irreducible WNR-surface germ, then its singular locus must be weakly normal, by case D. of (4.1.16).

(4.1.19) Remark : There are many cases not covered by (4.1.16). For example, the precise nature of the singular locus of the singularities as in (4.1.12) is unknown to me.

(4.1.20) About condition 2), the embedding of curve germs $\tilde{\Sigma}$ on a rational surface germ $\tilde{X}$ I do not have any general results. All I know is that *not all* curve germs can be embedded on a rational surface. In particular, by (4.1.17) not every irreducible Σ can occur as singular locus of WNR-germ X.

(4.1.21) Remember the equality $Mult(X,p) = Embdim(X,p) - 1$ of (4.1.6) for a WNR-surface germ X.
In general one has for a germ of an analytic space (X,p) the inequality:

$$Mult(X,p) \geq Embdim(X,p) - \dim(X,p) + 1$$

as one sees by taking repeated hyperplane sections. So WNR-surface germs are of *minimal multiplicity* (analogous to varieties of minimal degree).
Note the following trivial:

(4.1.22) **Lemma :** Let (X,Σ,p) be a WNCM-surface germ and let (C,p) be the germ of a general hyperplane section. Then:

$$\delta(C,p) \geq Mult(\Sigma,p)$$

proof : If we move the hyperplane defing C slightly, it cuts X in a curve C_t having $Mult(\Sigma,p)$ singular points. Now use the semicontinuity of the δ-invariant (2.1.7) ∎

This has an amusing consequence:

(4.1.23) **Corollary :** Let (X,Σ,p) be a WNR-surface germ. Then the embedding dimension of the singular curve Σ is *at least two less* than that of X (and usually much more than two).

proof :

$$\begin{aligned}
Embdim(X,p) &= Mult(X,p) + 1 && \text{by } (4.1.6)\\
Mult(X,p) &= \delta(C,p) + 1 && \text{by } (4.1.13)\\
\delta(C,p) &\geq Mult(\Sigma,p) && \text{by } (4.1.22)\\
Mult(\Sigma,p) &\geq Embdim(\Sigma,p) && \text{by } (4.1.21)
\end{aligned}$$

∎

(4.1.24) **Weakly rational double and triple points.**
The classification of non-isolated weakly rational singularities can be reduced to the (harder) problem of the classification of the isolated rational ones. The reason is that

the associated root graphs belonging to a WNR-surface germ X have to be graphs of rational singularities with the same multiplicity, because the fundamental cycle on a stable model is the fundamental cycle of the associated graph. So one can work backwards:

a. Look for *series* in the lists of isolated rational singularities.

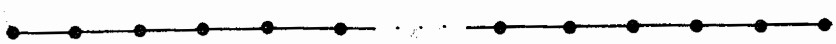
b. Try to interpret such a graph as a root graph associated to an improvement graph by declaring certain vertices to correspond to special roots. The negative definiteness of the improvement graph imposes conditions for this to be possible.

c. This way we will get *all* improvement graphs of non-isolated weakly rational surfaces.

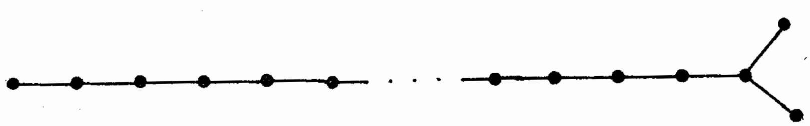
A. Weakly rational double points

The isolated ones were classified by DuVal [DuV].
It is the well-known A-D-E list:

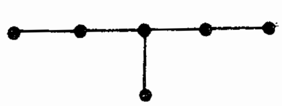
A_n



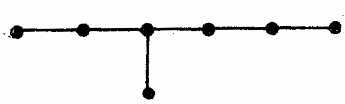
D_n



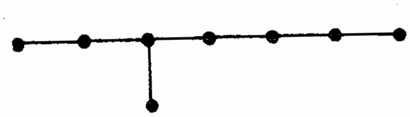
E_6



E_7

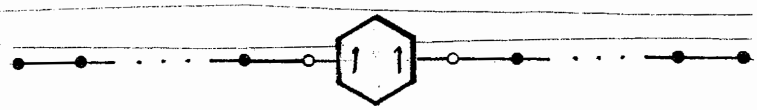


E_8

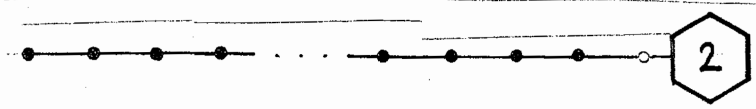


a. It is clear that only A_n and D_n come in series.

b. For A_n every vertex can correspond to a special root.



For D_n only the two end points connected to the branch point can correspond to special roots.



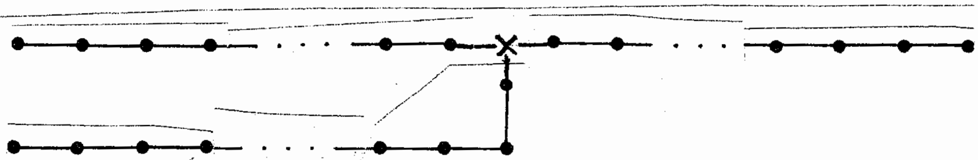
c. These improvement graphs define A_∞ and D_∞ .Hence: Classification of weakly rational double points (WRDP's)

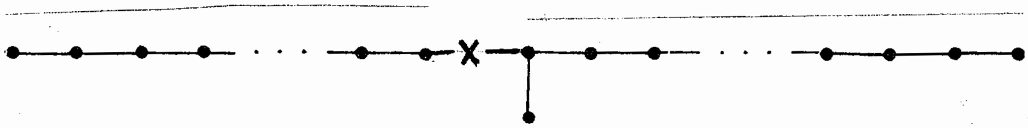
1) Isolated : A D E (the RDP's).

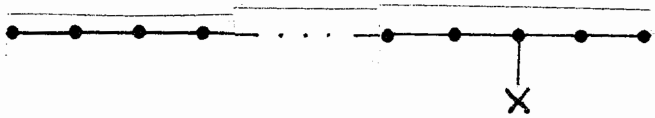
2) Non-isolated : A_∞ and D_∞ .

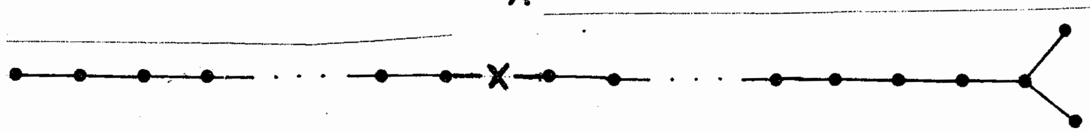
B. <u>Weakly rational triple points</u>

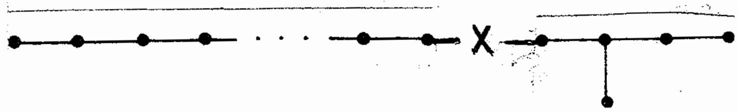
The isolated rational triple points were classified by Artin in [Art]. The list is less well known. (x denotes a (-3)-curve.)

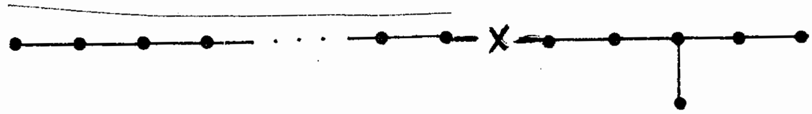
1. 

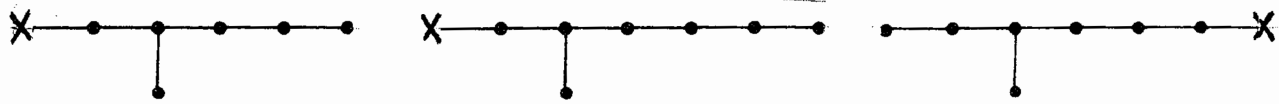
2. 

3. 

4. 

5. 

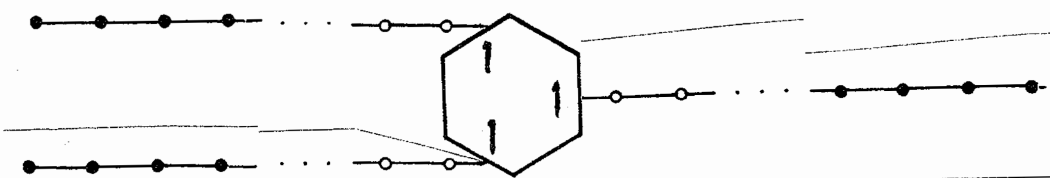
6. 

7. 

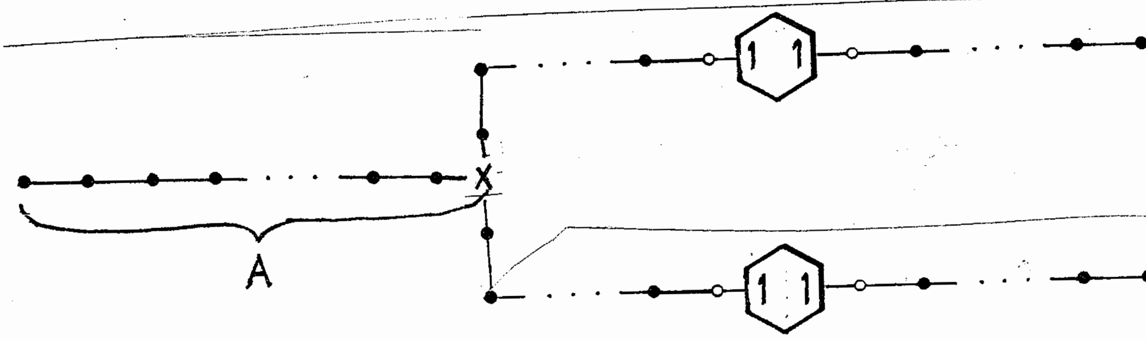
a. So we have six big series and three exceptional ones.

b. For graph 1. there are essentially three ways to declare vertices to correspond to special roots:

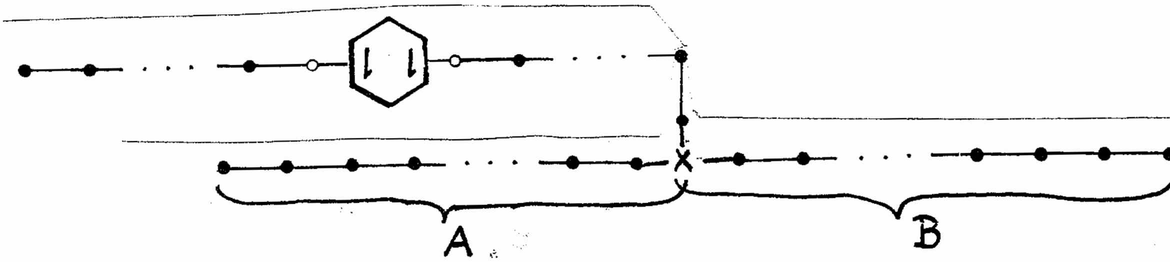
1



1(A)

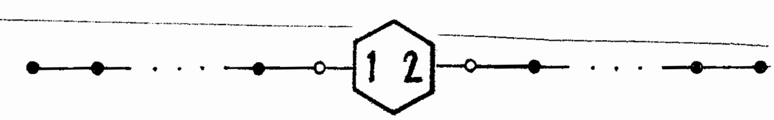


1(A,B)

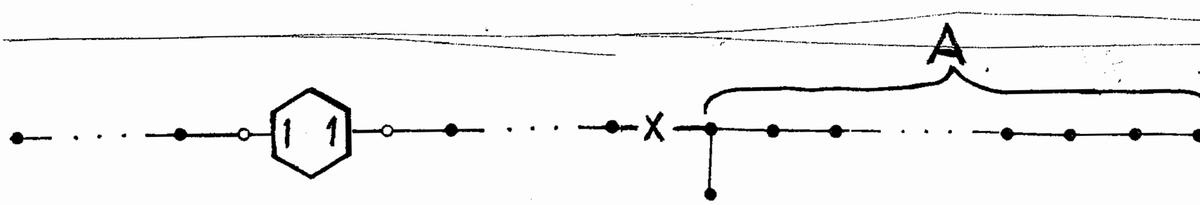


For graph 2. there are also three ways to assign special roots:

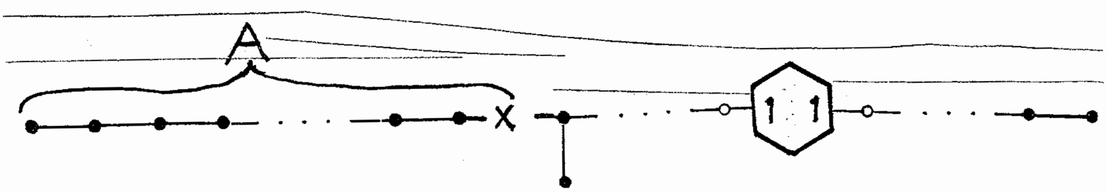
2



2(A)



2(A)*

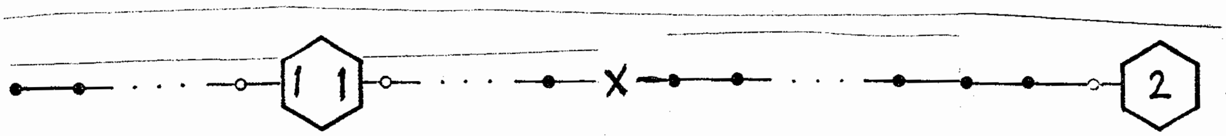


For graph 3. there is only one way to assign special roots:

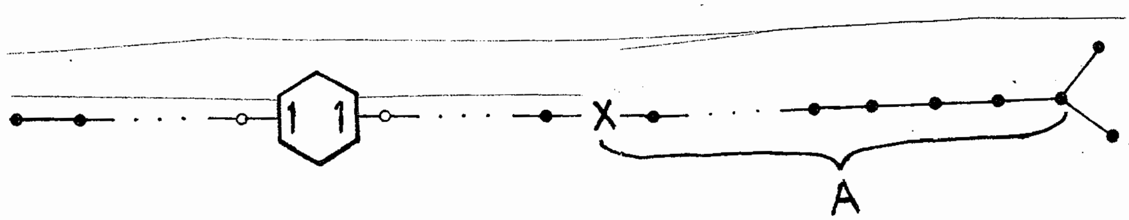
3

For graph 4. there are three ways to assign special roots:

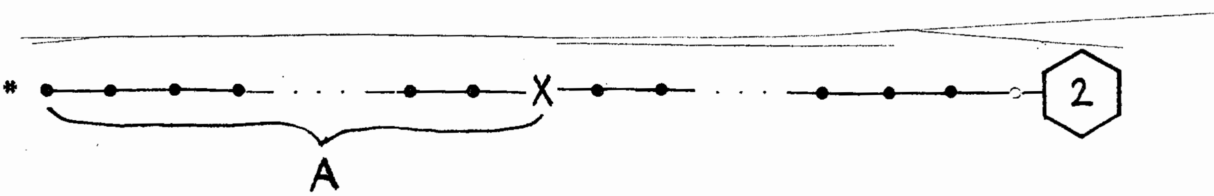
4



4(A)

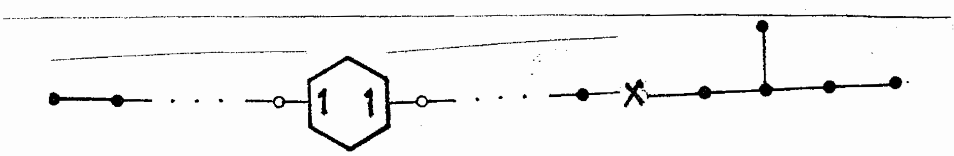


4(A)*

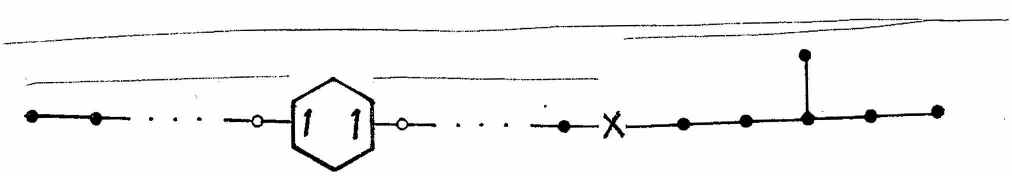


For graphs 5. and 6. there is essentially one way to assign a special root:

5



6



c. The resulting singularities can be interpreted as follows: First there are the partition singularities:

$$x_{(1,1,1)} \approx 1 \qquad x_{(2,1)} \approx 2 \qquad x_{(3)} \approx 3$$

Secondly there is the group of singularities obtained by glueing a smooth component to an WRDP along a smooth curve. Below we give a table of the WRDP's, together with a parametric representation of the curve and the resulting improvement graph.

WRDP	equation	curve	improvement
A_{a+b+1}	$x.y - z^{a+b}$	$(\lambda^a, \lambda^b, \lambda)$	1(A,B)
A_∞	$x.y$	$(\lambda^a, 0, \lambda)$	1(A)
D_{a+2}	$z^2 - y(x^2 - y^a)$	$\begin{cases}(\pm\lambda^{a/2}, \lambda, 0)\\(0, \lambda, \pm\lambda^{(a+1)/2})\end{cases}$	2(A)
		$(0, \lambda, 0)$	4(A)
D_∞	$z^2 - y.x^2$	$(0, \lambda, 0)$	4
E_6	$z^2 + y^3 - x^4$	$(\lambda, 0, \pm\lambda^2)$	5
E_7	$z^2 - y(x^3 - y^2)$	$(\lambda, 0, 0)$	6

Thirdly, there are two other series, not of the above types:

$2(A)^*$: Here X consists of two components. There is a smooth component, containing a curve with equation $y^2 = x^{2a+1}$, which is the singular locus. The other component is **not** Cohen-Macaulay. Its normalization is smooth and is glued to the first component via the normalization mapping of the curve. Example (1.1.16) is of this type.

$4(A)^*$: X is obtained by taking a standard line $(x = 0,\ z = 0)$ on an A_a singularity $(x.y = z^{a+1})$ and glueing this 2:1 to itself.

(4.1.25) Remark : Tjurina [Tj] has given equations for the rational triple points in the form of 2×2-minors of a 2×3-matrix. Of course, the equations for the non-isolated weakly rational triple points are obtained by setting certain terms of the entries of these matrices equal to zero. Alternatively, one can use the above geometrical description in combination with (1.2.9) to find equations.

§ 4.2 Minimally Elliptic Singularities

After treating the singularities with $p_g = 0$, should one proceed with the case $p_g = 1$? In some sense yes, but unfortunately the condition $p_g = 1$ cannot be read off from the resolution or improvement graph. Example (2.3.8) is an illustration of this phenomenon: a singularity with $p_g = 1$ and one with $p_g = 2$ and having the same improvement graph. But note that in this example the one with $p_g = 1$ is not Gorenstein. Laufer [Lau 3] discovered that the condition:

$$\text{"}p_g(X,p) = 1 \quad \text{and} \quad (X,p) \text{ is Gorenstein"}$$

can be effectively read off from the resolution graph. It turns out that this is also the case for WNCM-surfaces.

(4.2.1) Definition : Let $Y \xrightarrow{\pi} X$ be an improvement of a WNCM-surface germ (X,Σ,p).

A cycle $A \in C_Y^{>0}$ is called *elliptic* if and only if $\chi(\mathcal{O}_A) = 0$. It is called *minimally elliptic* if it is elliptic and for all $0 < B < A$ one has $\chi(\mathcal{O}_B) > 0$.

(4.2.2) Theorem ([Lau 3]) : Let $Y \xrightarrow{\pi} X$ be a weakly minimal stable model for X. Let Z be its fundamental cycle. Then the following conditions are equivalent:

1) Z is a minimally elliptic cycle.
2) For all $A \in C_Y$ one has $A.Z = - A.L_Y$ where L_Y is the cycle of (3.3.8)
3) $\chi(\mathcal{O}_Z) = 0$ and every proper subvariety of $E = \pi^{-1}(p)$ is the exceptional set of a weakly rational singularity.

The proof in the case of isolated singularities involves only formal manipulations with cycles and so continues to hold in this more general context.

(4.2.3) Definition : A singularity that fulfils one of the above equivalent conditions is called *minimally elliptic*.

(4.2.4) Corollary : The associated root graphs of a non-isolated minimally elliptic singularity are resolution graphs of isolated minimally elliptics. So one can start classifying using the lists of isolated minimally elliptic singularities, as given by Laufer [Lau 3]. This works in very much the same way as we did in § 4.1 with the rational singularities.

(4.2.5) Theorem : A WNCM-germ (X,Σ,p) is minimally elliptic if and only if

$$p_g(X,p) = 1 \quad \text{and} \quad (X,p) \text{ is Gorenstein}$$

proof : See [Lau 3], thm. 3.10 . By now the reader should be able to make the appropriate substitutions him or herself. But let us show "minimally elliptic $\Rightarrow$ Gorenstein". Let $Y \longrightarrow X$ be a weakly minimal stable model and Z its fundamental divisor. Look at the adjunction sequence (3.3.6)

$$0 \longrightarrow \omega_Y \longrightarrow \omega_Y(Z) \longrightarrow \omega_Z \longrightarrow 0$$

By Grauert-Riemenschneider (2.4.5) we get a surjection

$$H^0(\omega_Y(Z)) \longrightarrow H^0(\omega_Z)$$

As Z was by assumption an elliptic cycle, $\chi(\mathcal{O}_Z) = 1$, so $h^1(\mathcal{O}_Z) = 1$ and by duality $h^0(\omega_Z) = 1$. Let $\eta \in H^0(\omega_Y(Z))$ be an element not mapping to zero by the above surjection. The divisor of η can be written in the form $(\eta) = -Z + C + N$ where C has support on E and is ≥ 0 and N is non-compact ≥ 0. If $C > 0$, then η can be considered as a section in $H^0(\omega_Y(Z-C))$. But then we can form a commutative square:

$$\begin{array}{ccc} H^0(\omega_Y(Z-C)) & \longrightarrow & H^0(\omega_{Z-C}) \\ \downarrow & & \downarrow \\ H^0(\omega_Y(Z)) & \longrightarrow & H^0(\omega_Z) \end{array}$$

and by *minimal* ellipticity $H^0(\omega_{Z-C}) = 0$, so we conclude that $C = 0$. Using (4.2.2) 2) one then can show that $N = 0$. Hence we have constructed a dualizing differential η on $X - \{p\} \approx Y - E$, which is nowhere vanishing, so X is Gorenstein. (We have omitted some details.) ∎

(4.2.6) **Corollary :** On an improvement of a minimally elliptic singularity we only find the partition singularities A_∞ and D_∞ .

We conclude this brief detour into the minimally elliptic singularities by describing the structure of the non-isolated double and triple points.
The minimally elliptic double and triple points are hypersurfaces in $\mathbb{C}^3$. They can be divided in five groups.

Group 1. <u>The degenerate cusps</u>

This is a preferred class of singularities all appearing as deformations of $T_{\infty,\infty,\infty}$, the ordinary triple point. Their improvement graph is a cycle (Z is reduced) (see § 4.3).

We note that the general hyperplane section (through the singular point) of a WRDP of type D and E is a cusp (A_2), so has $\delta = 1$. This gives us two different ways to make a singularity with $p_g = 1$ out of them:

Group 2: <u>A*DE</u>

Here we take the WRDP of type D or E and form the union with its general hyperplane. For these singularities we have an A_∞ on the improvement.

Group 3: <u>D*DE</u>

Here we take the WRDP of type D or E and use the 2:1 map of the cusp to the line to glue. For these singularities we have a D_∞ on the improvement.
(When we do a similar operation on a WRDP of type A we end up with a degenerate cusp.)

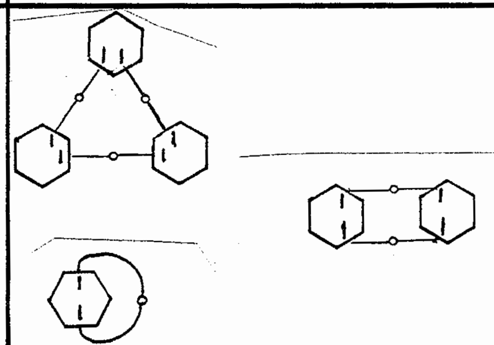
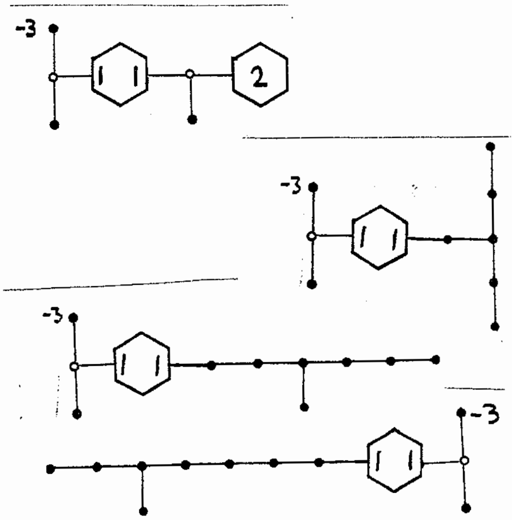
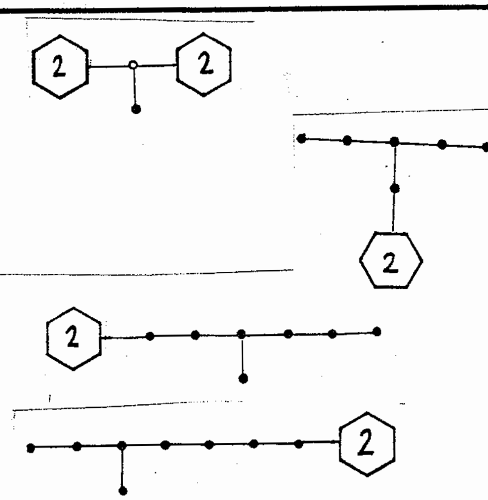
On certain rational triple points there are also curves with an A_2 singularity. These give in a similar way rise to:

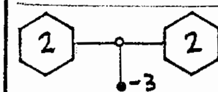
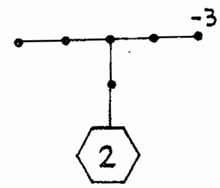
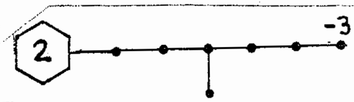
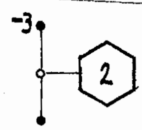
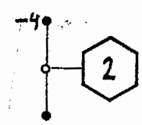
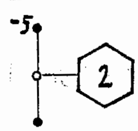
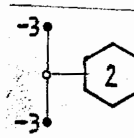
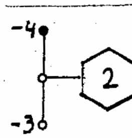
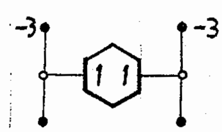
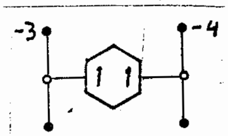
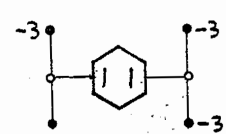
Group 4: <u>D*DE*</u>

Group 5. <u>The pyramid</u>

This is a group of eight singularities. They appear as the limits of the series of singularities U, S, W, Q, Z and J with $p_g = 1$ from the list of Arnol'd (see [Arn 1]).

We give a table of equations and improvement graphs for these singularities. (We only list the 'absolute limits'.)

Name	Equation	Improvement graph
$T_{\infty,\infty,\infty}$	$x.y.z$	
$T_{2,\infty,\infty}$	$z^2 + x^2.y^2$	
$J_{2,\infty}$	$z^2 + y^3 + x^2.y^2$	
$A{*}D_\infty$	$z.y^2 + z.x^3 + z^2.x^2$	
$A{*}E_6$	$z.y^2 + z.x^3 + z^5$	
$A{*}E_7$	$z.y^2 + z.x^3 + z^4.x$	
$A{*}E_8$	$z.y^2 + z.x^3 + z^6$	
$D{*}D_\infty$	$z^2 + y^3.x^2 + y^2.x^3$	
$D{*}E_6$	$z^2 + y^6 + y^2.x^3$	
$D{*}E_7$	$z^2 + y^5.x + y^2.x^3$	
$D{*}E_8$	$z^2 + y^7 + y^2.x^3$	

Name	Equation	Improvement graph
$D*D^{*}_{\infty}$	$z^2.x + z^2.y + y^2.x^2$	
$D*E^{*}_{6}$	$z^2.x + z.y^3 + y^2.x^2$	
$D*E^{*}_{7}$	$z^2.x + y^5 + y^2.x^2$	
$J_{3,\infty}$	$z^2 + y^3 + y^2.x^3$	
$Z_{1,\infty}$	$z^2 + y^3.x + y^2.x^3$	
$Q_{2,\infty}$	$z^2.x + y^3 + y^2.x^2$	
$W_{1,\infty}$	$z^2 + y^4 + y^2.x^2$	
$S_{1,\infty}$	$z^2.x + z.y^2 + y^2.x^2$	
$W^{\#}_{1,\infty}$	$z^2 + z.y^2 + z.x^3$	
$S^{\#}_{1,\infty}$	$z^2.x + z.y^2 + z.x^3$	
$U_{1,\infty}$	$z^3 + z.y^2 + z.x^3$	

§ 4.3 Gorenstein Du Bois surface singularities

Every space has attached to it its *filtered de Rham complex* $(\underline{\Omega}^{\cdot}_X, F^{\cdot})$, which plays from a Hodge theoretical point of view the same role as the ordinary de Rham complex on a smooth space (see [Stee 2], [Stee 3], [DuB]). The space X is called *Du Bois at $p \in X$* or (X,p) is a *Du Bois singularity* if and only if

(4.3.1)
$$\mathcal{O}_{X,p} \approx Gr^0_F(\underline{\Omega}^{\cdot}_X)_p$$

Such Du Bois singularities were studied in [Stee 2] and recently by Ishii in [I 1] and [I 2].

In this paragraph we are going to find all Gorenstein surface germs (X,p) which have this Du Bois property.

(4.3.2) **Lemma :** If (X,p) is Du Bois, then it is weakly normal. Let (X,p) be a WNCM-surface singularity and let $(Y,E) \xrightarrow{\pi} (X,p)$ be a good improvement of (X,p). Then (X,p) is Du Bois if and only if the restriction map

$$H^1(\mathcal{O}_Y) \longrightarrow H^1(\mathcal{O}_E)$$

is an isomorphism.

proof : For the first statement see [Stee 5]. The second statement follows from [I 1], prop 1.4 , and is due to the fact that the complex $(\underline{\Omega}^{\cdot}_X, F)$ has a glueing property. ∎

Define $\omega_Y(E) := \mathcal{H}om(\mathcal{J}, \omega_Y)$, where $\mathcal{J}$ is the ideal sheaf of E in Y. This sheaf sits in an exact sequence:

$$0 \longrightarrow \omega_Y \longrightarrow \omega_Y(E) \longrightarrow \omega_E \longrightarrow 0$$

(see § 2.4). But note that even if Y is Gorenstein (only A_∞ and D_∞ points on Y) the sheaf $\omega_Y(E)$ is not locally free at the points where E is not Cartier. On a general point of E however it coincides with the sheaf of two-forms on Y with a simple pole along E and it is the unique reflexive sheaf with this property.

We now give a dual version of the Du Bois property which gives a bound on the pole order of dualizing differentials on Y-E.

(4.3.3) Lemma : Let $(Y,E) \longrightarrow (X,p)$ be a good improvement of a WNCM-surface germ. Then (X,p) is Du Bois if and only if the natural map

$$H^0(\omega_Y(E))/H^0(\omega_Y) \longrightarrow H^0(\omega_{Y-E})/H^0(\omega_Y)$$

is an isomorphism.

proof : (c.f. [I 2], thm. 1.8) By Grauert-Riemenschneider (2.4.5) we have that

$$H^0(\omega_Y(E))/H^0(\omega_Y) \approx H^0(\omega_E)$$

Duality isomorphism on E: $H^0(\omega_E)^* = H^1(\mathcal{O}_E)$

By the Du Bois property : $H^1(\mathcal{O}_Y) \approx H^1(\mathcal{O}_E)$

Duality isomorphism on Y: $H^1(\mathcal{O}_Y)^* = H^1_E(\omega_Y)$

So the result follows by using Grauert-Riemenschneider again. ∎

(4.3.4) Corollary : Let $(Y,E) \xrightarrow{\pi} (X,p)$ be a good improvement of a Gorenstein Du Bois surface germ. Let K_Y be the canonical cycle as in (3.3.14). Then one has:

$$K_Y \geq -E$$

By (3.3.22) we know that on a *weakly minimal* model K_Y has full support (X is CM, so Y is connected) or is zero. From this it follows that there are two possibilities:

A) $K_Y = 0$ ➡ X is an R.D.P , A_∞ or D_∞ (so X is a WRDP).

B) $K_Y = -E$ ➡ E is Cartier ➡ Y contains no D_∞ points.

In case B) one deduces: $\omega_Y = \mathcal{O}_Y(-E)$ so by the usual adjunction sequence for E we get:

$$\omega_E \approx \mathcal{O}_E$$

(4.3.5) Lemma : Let E be a reduced curve with at most ordinary double points as singularities and with the property that $\omega_E \approx \mathcal{O}_E$. Then one has the following possibilities for E:

1) E is a smooth elliptic curve.

2) E is a cycle of rational curves. ($E = \bigcup_{i=1}^{r} E_i$, $E_i . E_{i+1} = 1$, E_i smooth rational curves. r=1 is allowed.)

proof: Well-known, see [Do]. ∎

With the help of the above lemma we find the a priori possibilities for the exeptional sets of Gorenstein Du Bois singularities. Any improvement Y which has as exceptional divisor a curve E as in (4.3.4) defines a minimally elliptic singularity (with reduced fundamental cycle), by (4.2.2). Hence it has $p_g = 1$ and so it is Du Bois. We can conclude:

(4.3.6) Theorem : Let (X,p) be a Gorenstein Du Bois surface germ. Then one has the following possibilities:

1) $p_g(X,p) = 0$ <=> X is a WRDP

2) $p_g(X,p) = 1$ <=> X is minimally elliptic with reduced fundamental cycle

In other words:

A) p isolated singular point

Then X is one of the following:
* RDP
* Simple elliptic
* Cusp singularity

B) p non-isolated singular point

Then X is one of the following:
* A_∞ , D_∞
* Degenerate cusp

(4.3.7) Remark : Part A of (4.3.5) is contained in [Stee 3]. In the same article, part B is posed as a question of N. Shepherd Barron. For the definition and some properties of degenerate cusps see [Sh].

§ 4.4 The First Betti Number of a Smoothing

The following result is due to Greuel & Steenbrink:

(4.4.1) Theorem : Let $\mathcal{X} \xrightarrow{f} S$ be a smoothing of a *normal isolated* singularity $X = f^{-1}(0)$. Let $X_t = f^{-1}(t)$, $t \neq 0$, be its Milnor fibre. Then:

$$b_1(X_t) := \dim_{\mathbb{C}} H^1(X_t, \mathbb{C}) = 0$$

For a proof see [G-S].

When one looks for a similar simple statement for non-isolated singularities one runs soon into big trouble. By taking the cone over Zariski's plane sextic with six cusps, we get a surface in $\mathbb{C}^3$. The first Betti number of the Milnor fibre of this surface (which thus appears as a six-fold cover of the complement of the curve) depends on the position of the cusps: when they are on a conic, then $b_1(X_t) = 2$, when they are not, then $b_1(X_t) = 0$. (see [Es]). This shows that b_1 is a subtle invariant.

The cone over a curve $\Gamma \subset \mathbb{P}^2$ is weakly normal precisely when Γ has only ordinary double points. In that case the first Betti number is independent of the exact position of the double points: one has $b_1(X_t) = r - 1$, where r is the number of irreducible components of Γ. We are going to proof the following generalization of theorem (4.4.1):

Theorem : Let $\mathcal{X} \xrightarrow{f} S$ be a smoothing of a (reduced, equidimensional and) *weakly normal* space (germ) X. Let $X_t = f^{-1}(t)$, $t \neq 0$, be its Milnor fibre and r the number of irreducible components of X. Then:

$$b_1(X_t) \leq r - 1$$

For a hypersurface one has equality.

The proof will be along the lines of [G-S].

(4.4.2) Notation & Topological description

Let X be a fixed contractible Stein representative of a *reduced* and *equidimensional* germ (X,p).
We consider a smoothing of X over a smooth curve (germ) S:

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \\ \{0\} & \longrightarrow & S \end{array}$$

We also assume that $\mathcal{X}$ is contractible and Stein. Remark that in this situation we have that $\mathcal{X}$ is *normal*: $\mathrm{Sing}(\mathcal{X}) \subset \Sigma := \mathrm{Sing}(X)$, so this is of codimension ≥ 2. Further $\mathrm{depth}_{\Sigma}(X) \geq 1$, so we have $\mathrm{depth}_{\Sigma}(\mathcal{X}) \geq 2$.

To study the Milnor fibre $X_t := f^{-1}(t)$, $t \neq 0$, it is convenient to take an *embedded resolution* of X in $\mathcal{X}$. So we get a space $\mathcal{Y}$ together with a proper map $\mathcal{Y} \xrightarrow{\pi} \mathcal{X}$ with the following properties:

1) $\mathcal{Y} - \pi^{-1}(\Sigma) \longrightarrow \mathcal{X} - \Sigma.$
2) $Y := (f \circ \pi)^{-1}(0)$ is a normal crossing divisor.
3) $\mathcal{Y}$ is smooth.

After a finite base change we may assume that Y is reduced.
(Semi-stable reduction.)
In Y we find in general three types of divisors:

a) $\tilde{X}$, the strict transform of X.
b) F, a set of non-compact divisors, mapping properly to Σ.
c) $E = \pi^{-1}(p)$, a compact divisor.

(c.f. with the situation in (2.6.?)).

The Milnor fibre X_t is via π isomorphic to $Y_t := (f \circ \pi)^{-1}(t) \subset \mathcal{Y}$.
In a semi-stable family this Milnor fibre Y_t "passes along" every component of Y just once. One can find a "contraction"

$$c: Y_t \longrightarrow Y$$

of the Milnor fibre Y_t on the special fibre Y (see [Cl],[Stee 1]).
Now we can use the Leray spectral sequence for c to find the beginning of an exact sequence:

Leray:

$$0 \longrightarrow H^1(Y) \longrightarrow H^1(Y_t) \longrightarrow H^0(\mathbb{C}_{Y^{[0]}}/\mathbb{C}_Y) \longrightarrow H^2(Y) \longrightarrow$$

Here we have used the easily verified formulas:

$$c_*\mathbb{C}_{Y_t} = \mathbb{C}_Y$$
$$R^1c_*\mathbb{C}_{Y_t} = \mathbb{C}_{Y^{[0]}}/\mathbb{C}_Y$$

$Y^{[0]} := \coprod Y_i$, where Y_i are the irreducible components of Y. (The sheaf $\mathbb{C}_{Y^{[0]}}$ is considered on Y.)

We note that there are two other exact sequences in which $H^1(Y_t)$ appears:

Milnor's Wang sequence (see [Mi], p. 67)

$$0 \longrightarrow H^0(Y_t) \longrightarrow H^1(B) \longrightarrow H^1(Y_t) \xrightarrow{h_*-Id} H^1(Y_t) \longrightarrow \cdots$$

Here $B = \mathcal{X} - X$, the total space of the Milnor fibration over $S - \{0\}$ and h_* is the monodromy transformation.

Sequence of the pair $B = \mathcal{Y} - Y \longrightarrow \mathcal{Y} \approx Y$($\approx$ means homotopy equivalence)

$$0 \longrightarrow H^1(Y) \longrightarrow H^1(B) \xrightarrow{\alpha} H^0(Y^{[0]}) \xrightarrow{\beta} H^2(Y) \longrightarrow \cdots$$

Here we have used the isomorphism $H^2(\mathcal{Y},\mathcal{Y} - Y) \approx H^0(Y^{[0]})$

These three sequences fit into a single big diagram:

$$\begin{tikzcd}
 & & \vdots & \vdots & & \\
 & & H^1(Y_t) \arrow[u] & H^1(Y) \arrow[u] & & \\
0 \arrow[r] & H^1(Y) \arrow[r] & H^1(Y_t) \arrow[r] \arrow[u, "h_*-id"'] & H^0(\mathbb{C}_{Y^{[0]}}/\mathbb{C}_Y) \arrow[r] \arrow[u] & H^2(Y) \arrow[r] & {} \\
0 \arrow[r] & H^1(Y) \arrow[r] \arrow[u] & H^1(B) \arrow[r, "\alpha"] \arrow[u] & H^0(Y^{[0]}) \arrow[r, "\beta"] \arrow[u] & H^2(Y) \arrow[r] \arrow[u] & {} \\
 & & H^0(Y_t) \arrow[r] \arrow[u] & H^0(Y) \arrow[u] & &
\end{tikzcd}$$

We note that dim $H^0(Y) = 1$.

From this diagram we draw the following conclusions:

(4.4.3) Conclusion: In the above sitution we have:

1) $\dim H^1(B) = \dim H^1(Y) + \dim \ker \beta$.

2) $\dim H^1(Y_t) \geq \dim H^1(B) - 1$, and equality holds if $H^1(Y) = 0$.

3) If $H^1(Y) = 0$, then the monodromy acts trivially on $H^1(Y_t)$.

We now study the parts $H^1(Y)$ and $\ker \beta$ separately.

(4.4.4) The group $H^1(Y)$. If X is a plane curve singularity, then one can compute $\dim H^1(Y)$. The result is:

$$\dim H^1(Y) = 2.g + b$$

where g is the sum of the genera of the compact components of Y and b is the number of cycles in the dual graph of Y. (These numbers g and b are invariants of the limit Mixed Hodge Structure on $H^1(X_t)$; one has $b = \dim Gr_0^W Gr_F^1 H^1(Y_t)$, $g = \dim Gr_1^W Gr_F^1(Y_t)$, see [Stee 1].). By taking $X \times \mathbb{C}$ we can construct (trivial) examples of *irreducible* surfaces with $H^1(X_t)$ arbitrarily high. Only in the case that X is an ordinary double point, one has $H^1(Y) = 0$. It turns out that it is exactly the *weak normality of X_0* that forces $H^1(Y)$ to vanish.

(4.4.5) Proposition : Let $\mathcal{X} \xrightarrow{f} S$ be a flat deformation of a weakly normal $X = f^{-1}(0)$. Let $\mathcal{Y} \xrightarrow{\pi} \mathcal{X}$ be map such that:

1) $\mathcal{Y} - \pi^{-1}(\Sigma) \longrightarrow \mathcal{X} - \Sigma$, $\Sigma = \mathrm{Sing}(\Sigma)$

2) $\pi_* \mathcal{O}_{\mathcal{Y}} \approx \mathcal{O}_{\mathcal{X}}$.

Then one has: $$R^1\pi_*\mathcal{O}_{\mathcal{Y}} = 0.$$

proof : This is the crucial point and the argument is the same as in [G-S]. First look at the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{Y}} \xrightarrow{t} \mathcal{O}_{\mathcal{Y}} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

Here t is a local parameter on S and Y is the fibre over 0. Taking the direct image of the above sequence gives a diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \pi_*\mathcal{O}_{\mathcal{Y}} & \xrightarrow{t} & \pi_*\mathcal{O}_{\mathcal{Y}} & \longrightarrow & \pi_*\mathcal{O}_Y & \longrightarrow & R^1\pi_*\mathcal{O}_{\mathcal{Y}} & \xrightarrow{t} & R^1\pi_*\mathcal{O}_{\mathcal{Y}} \longrightarrow \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{O}_{\mathcal{X}} & \xrightarrow{t} & \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 & &
\end{array}$$

By assumption $\pi_*\mathcal{O}_{\mathcal{Y}} \approx \mathcal{O}_{\mathcal{X}}$. From this it follows that the sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_Y \longrightarrow R^1\pi_*\mathcal{O}_{\mathcal{Y}} \xrightarrow{t} R^1\pi_*\mathcal{O}_{\mathcal{Y}}$$

is also exact. We claim that $\mathcal{O}_X \approx \pi_*\mathcal{O}_Y$. Note that by condition 2) we have that the fibres of π are *connected*. Consider a section $g \in \pi_*\mathcal{O}_Y$, or what amounts to the same, a funcion on Y. As the π-fibres are compact and connected, this function is *constant* along the π-fibres. Hence g can be considered as a *continous* function on X , which is holomorphic on $Y - \pi^{-1}(\Sigma) \longrightarrow X - \Sigma$. Because we assumed X to be weakly normal, $g \in \mathcal{O}_X$. So we have indeed $\mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_Y$. Because the map π is an isomorphism outside Σ, the coherent sheaf $R^1\pi_*\mathcal{O}_{\mathcal{Y}}$ has as support a set contained in Σ. By the last exact sequence t acts *injectively* $R^1\pi_*\mathcal{O}_{\mathcal{Y}}$. As t vanishes on Σ ($\subset$ X) we conclude that $R^1\pi_*\mathcal{O}_{\mathcal{Y}} = 0$. ∎

(4.4.6) Corollary : Let C be a weakly normal curve singularity and X the total space of a flat deformation $X \longrightarrow S$ of C. Then X is weakly rational. This was stated as (2.5.7).

(4.4.7) Proposition : With the notation of (4.4.2) we have:

$$X \text{ weakly normal} \Rightarrow H^1(Y) = 0$$

proof : The embedded resolution map $\mathcal{Y} \longrightarrow \mathcal{X}$ clearly fulfils condition 1) of (4.4.5). It also fulfils condition 2), because $\mathcal{X}$ is normal, hence $\mathcal{O}_{\mathcal{X}} \longrightarrow i_*\mathcal{O}_{\mathcal{X}-\Sigma}$ where $i: \mathcal{X} - \Sigma \longrightarrow \mathcal{X}$ is the inclusion map. Because $\mathcal{Y} - \pi^{-1}(\Sigma) \longrightarrow \mathcal{X} - \Sigma$, it follows that $\pi_*\mathcal{O}_{\mathcal{Y}} \approx \mathcal{O}_{\mathcal{X}}$. Now according to (4.4.5) we have $R^1\pi_*\mathcal{O}_{\mathcal{Y}} = 0$, in other words :

$$H^1(\mathcal{O}_{\mathcal{Y}}) = 0$$

From the exponential sequence

$$0 \longrightarrow \mathbb{Z}_{\mathcal{Y}} \longrightarrow \mathcal{O}_{\mathcal{Y}} \longrightarrow \mathcal{O}^*_{\mathcal{Y}} \longrightarrow 0$$

and the similar sequence for $\mathcal{X}$ and the fact that $\mathcal{O}_{\mathcal{X}} \approx \pi_* \mathcal{O}_{\mathcal{Y}}$ it then follows that:

$$H^1(\mathcal{Y}, \mathbb{Z}_{\mathcal{Y}}) = 0$$

As $\mathcal{Y}$ is contractible to Y, we have $H^1(Y,\mathbb{Z}) = 0$ ∎

(4.4.8) The kernel of β.

In the big diagram of (4.4.2) there was a map β

$$H^0(Y^{[0]},\mathbb{Z}) \xrightarrow{\beta} H^2(\mathcal{Y},\mathbb{Z}) \quad (= H^2(Y,\mathbb{Z}))$$

This map works as follows: Elements of the first groups can be considered as *divisors* $\sum a_i.Y_i$ with support on Y. (The Y_i are the irreducible components of Y.) Then one has:

$\beta(\sum n_i.Y_i)$ = first chern class of the line bundle determined by the divisor $\sum n_i.Y_i$

So the map β factorizes over the map ψ which associates to a divisor its line bundle:

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(\mathcal{O}_{\mathcal{Y}}) \arrow[r] & H^1(\mathcal{O}^*_{\mathcal{Y}}) \arrow[r] & H^2(\mathcal{Y},\mathbb{Z}) \arrow[r] & \cdots \\
& & H^0(Y^{[0]},\mathbb{Z}) \arrow[u, "\psi"] \arrow[ur, "\beta"'] & &
\end{tikzcd}$$

We first study the map ψ. Note that if $H^1(\mathcal{O}_{\mathcal{Y}}) = 0$, then we have $\ker \psi = \ker \beta$.

(4.4.9) Definition : Let $(\mathcal{X},p)$ be a germ of a *normal* analytic space. The *local class group* is the group

$$Cl_p(X) = \{(\text{germs of}) \text{ Weil divisors}\}/\{(\text{germs of}) \text{ principal divisors}\}$$

(4.4.10) Proposition : With the notation as in (4.4.2) there is a diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \ker\psi & \longrightarrow & \ker\gamma & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(F^{[0]}) & \longrightarrow & H^0(Y^{[0]}) & \longrightarrow & H^0(X^{[0]}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & H^0(F^{[0]}) & \longrightarrow & H^1(\mathcal{O}^*_{\mathcal{Y}}) & \longrightarrow & Cl_p(\mathcal{X}) & \longrightarrow & 0
\end{array}$$

Here $F^{[0]} = Y^{[0]} \setminus \tilde{X}$ (so it contains the components F and E of (4.4.2)) and $X^{[0]} = \coprod X_i$, where the X_i are the irreducible components of X. The maps are the obvious ones.

proof : The surjection of $H^1(\mathcal{O}^*_{\mathcal{Y}})$ (or better of $R^1\pi_*\mathcal{O}^*_{\mathcal{Y}}$) to the local class group is obvious: pulling back a Weil divisor on $\mathcal{X}$ gives a Cartier divisor on $\mathcal{Y}$ (hence a line bundle) that maps down to the original Weil divisor, as the map π is a modification in codimension ≥ 2 (c.f. [Mu]). The main point is to show that the kernel of the bottom row is not bigger, or what amounts to the same, that $\ker\psi \approx \ker\gamma$. Let $A = \sum a_i.Y_i$ be in the kernel of ψ. We may assume that $a_i \geq 0$. Hence there is a function $g \in H^0(\mathcal{O}_{\mathcal{Y}})$ with $(g) = A$. By the normality of $\mathcal{X}$ we have $\mathcal{O}_{\mathcal{X}} = \pi_*\mathcal{O}_{\mathcal{Y}}$, so g can be considered as holomorphic on $\mathcal{X}$, having of course as divisor on $\mathcal{X}$ just (the image) of that part of A that does not involve $F^{[0]}$. This gives the map $\ker\psi \longrightarrow \ker\gamma$. This map is injective because if the divisor of g (on $\mathcal{X}$) would be zero, g would be a unit, hence A = 0. Surjectivity follows by pulling back functions. ∎

The use of (4.4.10) is that we get rid of the global object $\mathcal{Y}$ In (4.4.2) we used a base change to arrive at a semi-stable family. The kernel of the map γ is essentially independent of this base change:

(4.4.11) Lemma : Consider a normal space $\mathcal{X}$ and a reduced principal divisor $X \subset \mathcal{X}$. Let $X^{[0]} = \coprod X_i$, where the X_i are the irreducible components of X. Let $\tilde{\mathcal{X}}$ be obtained from $\mathcal{X}$ by taking a d-fold cyclic cover branched along X. Let $\gamma : H^0(X^{[0]}) \longrightarrow Cl_p(\mathcal{X})$, $\tilde{\gamma} : H^0(X^{[0]}) \longrightarrow Cl_p(\tilde{\mathcal{X}})$ be the obvious maps. Then $\ker(\gamma) \otimes \mathbb{Q} = \ker(\tilde{\gamma}) \otimes \mathbb{Q}$

proof : Excercise. ∎

We summarize the above results in one theorem:

(4.4.12) Theorem : Let $\mathcal{X} \xrightarrow{f} S$ be (a contractible Stein representative of) a smoothing of a reduced germ (X,p). Let $X_t = f^{-1}(t)$, $t \neq 0$, be its Milnor fibre.
Let $X^{[0]} = \coprod X_i$, where the X_i are the irreducible components.
Let $\gamma : H^0(X^{[0]}) \longrightarrow Cl_p(\mathcal{X})$ be the obvious map.
Then one has:

1) $b_1(X_t) \geq \text{rank } (\ker \gamma) - 1$.

2) If X is weakly normal, then one has equality:

$$b_1(X_t) = \text{rank } (\ker \gamma) - 1.$$

In particular, when X is a *hypersurface*, rank ker γ is equal to the number of irreducible components of X.

(4.4.13) Remark : For a hypersurface germ X in $\mathbb{C}^3$ with a *complete intersection* as singular locus and transversal type A_1 it is known that the first Betti number $b_1(X_t)$ is zero or one (see [Sie 2], [Str]). So the number of irreducible components of X is one or two. To put it in another way, the singular locus of a weakly normal hypersurface in $\mathbb{C}^3$ which has more than three components is *never* a complete intersection.

(4.4.14) Question : J. Stevens has shown that all degenerate cusps are smoothable (private communication). What is the first Betti number for these smoothings? Is the first Betti number an invariant of X? (Probably not, but at this moment I do not have computed any non-trivial example.)

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INDEX

SAMENVATTING

Dit proefschrift heeft een speciale klasse van complexe oppervlak singulariteiten tot onderwerp, te weten de klasse van de *zwak normale (Cohen-Macaulay)* oppervlak singulariteiten. Deze singulariteiten kunnen bestudeerd worden met methoden die analoog zijn met die welke doorgaans voor normale oppervlak singulariteiten gebruikt worden. Van fundamenteel belang is het begrip *verbetering* van een zwak normale oppervlak singulariteit. Dit begrip werd door N. Shepherd Barron ingevoerd en is het analogon van het begrip resolutie van een normale oppervlak singulariteit. Een verbetering is in tegenstelling tot een resolutie niet glad, maar bevat in het algemeen zogenaamde *partitie singulariteiten*. Met behulp van een verbetering kan men een invariant van een zwak normale oppervlak singulariteit invoeren die het *meetkundig geslacht* genoemd wordt. Een belangrijke eigenschap van deze invariant is zijn boven half continuiteit onder platte deformaties over een gladde basis kromme. Op een verbetering kan men op de gebruikelijke wijze met één-cykels rekenen, maar om tot een bevredigende theorie van de *fundamentele cykel* te komen is het in het algemeen noodzakelijk om door middel van opblazen naar een zogenaamd *stabiel model* over te gaan. Van belang is het daarbij behorende *wortel rooster.* Het vermoeden wordt uitgesproken dat de bij het wortel rooster behorende graaf een resolutiegraaf is van een normale oppervlak singulariteit. Wanneer de cykel theorie ver genoeg ontwikkeld is, kan men eenvoudig de reeds bekende resultaten over rationale en minimaal elliptische singulariteiten generaliseren tot zwak normale oppervlak singulariteiten. Hierbij dient opgemerkt te worden dat niet-geïsoleerde zwak normale Cohen-Macaulay oppervlak singulariteiten met meetkundig geslacht gelijk aan nul *niet* rationaal zijn in de gebruikelijke zin van het woord.

Het blijkt dat irreducibele zwak normale ruimten de eigenschap hebben dat het eerste Betti getal van de vezel in een vergladding nul is. Dit resultaat was al eerder bekend voor normale geïsoleerde singulariteiten.

CURRICULUM VITAE

De schrijver van dit proefschrift werd op 10 oktober 1958 in Amsterdam geboren. In 1977 slaagde hij voor het V.W.O. eindexamen aan het Gemeentelijk Atheneum in Utrecht. In datzelfde jaar begon hij met de studie natuurkunde aan de Rijks Universiteit van Utrecht. In 1979 werd het kandidaatsexamen wiskunde afgelegd. In december 1982 studeerde hij cum laude af in de wiskunde met als bijvak theoretische natuurkunde. In de daar opvolgende vier jaar verrichtte hij in dienst van de stichting Zuiver Wetenschappelijk Onderzoek promotie-onderzoek op het gebied van de singulariteiten theorie aan het Mathematisch Instituut van de Rijks Universiteit van Leiden onder leiding prof. dr. J.H.M. Steenbrink. Thans is hij verwikkeld in een dienstweigeringsprocedure.